REPRESENTATIONS OF WREATH PRODUCTS

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Reference:
A. Khare, *Functoriality of the BGG Category* $\mathcal{O}$,
`http://www.math.ucr.edu/~apoorva/job/skew.pdf`

0.1. The setup.

We will always work over $\mathbb{C}$.
Fix a complex semisimple Lie algebra $\mathfrak{g}$ and $n \in \mathbb{N}$.
Let $A = \mathfrak{U}(\mathfrak{g}^{\oplus n}) = (\mathfrak{U}\mathfrak{g})^{\otimes n}$.
$\Gamma = S_n$ acts on A by permuting the factors.
Let R_n be the *wreath product algebra* $S_n \wr \mathfrak{U}\mathfrak{g} = A \rtimes \Gamma$.

0.2. **Problems.**

$(A = \mathfrak{U}(\mathfrak{g}^{\oplus n}),\ \Gamma = S_n,\ R_n = A \rtimes \Gamma = S_n \wr \mathfrak{U}\mathfrak{g}.)$

1. Does complete reducibility hold for finite-dimensional R_n-modules?
2. What is the character of a given finite-dimensional (simple) R_n-module?
3. What is the center of R_n, and the set of central characters?
4. Is there a "direct sum decomposition" for R_n-modules using central characters, and does it hold for more R_n-modules than finite-dimensional ones?

We know all answers when $n = 1$, or for A-modules (i.e. set $\Gamma = 1$).

0.3. **Variants.**

What if we replace Γ by some subgroup of S_n, and A by $A_q := (U_q(\mathfrak{g}))^{\otimes n}$, or $A_\kappa := \mathcal{H}_\kappa^{\otimes n}$ (*infinitesimal Hecke algebra* over $\mathfrak{sl}_2$ - K.; Etingof, Ginzburg and Gan), or its quantized analogue $A_{q,\kappa}$ (Gan, K.)?

Examples that can be analyzed:

- $U_q(\mathfrak{g})^{\otimes n} \rtimes (\mathbb{Z}/n\mathbb{Z})$ (the group acts by permutations).
- $\big(U_q(\mathfrak{g})^{\otimes n} \otimes \mathfrak{U}\mathfrak{g}^{\otimes m}\big) \rtimes (S_n \times S_m)$ (the groups act on the respective factors).
- $\otimes_{i=1}^n (S_{n_i} \wr \mathfrak{U}\mathfrak{g}_i)$.
- $\otimes_{i=1}^n U_q(\mathfrak{g}_i)$ (set all $\Gamma = 1$).

The above questions can be answered in a unified setup.

In all cases, the problem reduces to studying modules over finite-dimensional algebras.

0.4. **Complete reducibility.**

This holds in vast generality: given an algebra A and a finite group Γ acting on it,

$$\mathrm{Hom}_{A\rtimes\Gamma}(M,-)=\left(\mathrm{Hom}_A\left(\mathrm{Res}_{\mathrm{A}}^{\mathrm{A}\rtimes\Gamma}M,\mathrm{Res}_{\mathrm{A}}^{\mathrm{A}\rtimes\Gamma}-\right)\right)^{\Gamma}$$

In our case, all functors are exact by Weyl and Maschke's Theorems, so complete reducibility holds.

0.5. **Characters of finite-dimensional modules.**

Each finite-dimensional R_n-module is an $A = \mathfrak{U}(\mathfrak{g}^{\oplus n})$-module, hence H-semisimple, where $H = \mathfrak{U}(\mathfrak{h}^{\oplus n})$.

Theorem 1. *Suppose V is such a module, simple; let v_λ be any highest weight vector in V. (Note: $\lambda = (\lambda_1, \dots, \lambda_n) \in H\widehat{\ } := \left(\mathfrak{h}^{\oplus n}\right)^*$ is dominant integral.)*

(1) *Define $M := \mathbb{C}[S_n]v_\lambda$. Then M is a finite-dimensional simple module over $H \rtimes S_n = S_n \wr \mathfrak{U}\mathfrak{h}$.*

(2) *M is the set of all A-maximal vectors: $M = \ker N_+ \subset V$, where $N_+ := \mathfrak{n}_+^{\oplus n} \cdot \mathfrak{U}(\mathfrak{n}_+^{\oplus n})$.*

(3) *If $ch_M = \sum_{\sigma \in S_n/S_n^\lambda} m_\sigma \sigma(\lambda)$, then as A-modules,*

$$V \cong \bigoplus_{\sigma \in S_n/S_n^\lambda} L_A(\sigma(\lambda))^{\oplus m_\sigma} \tag{1}$$

Remarks.

(1) $H\hat{}$ is an abelian group because H is a cocommutative Hopf algebra.

(2) $L_A(\lambda)$ is the finite-dimensional simple $\mathfrak{U}(\mathfrak{g}^{\oplus n})$-module with highest weight $\lambda \in H\hat{}$. (If λ is dominant integral, then so is $\sigma(\lambda)$ for all $\sigma \in S_n$.)

(3) Formal characters can now be computed using the Weyl Character Formula.

(4) The result holds for all λ, not only dominant integral ones. To see this, we work (later) inside the *BGG Category* $\mathcal{O}$.

0.6. **Center and central characters.**

Theorem 2.

(1) $\mathfrak{Z}(A \rtimes \Gamma) = \mathfrak{Z}(A)^\Gamma = S_n$*-invariants of* $(\mathfrak{Z}(\mathfrak{U}\mathfrak{g}))^{\otimes n} \cong \mathbb{C}[X_1, \ldots, X_s]$, *for* $s = n \cdot rk(\mathfrak{g}) = \dim_{\mathbb{C}}(H\widehat{\ })$.

(2) $\mathrm{Specm}(\mathfrak{Z}(A \rtimes \Gamma)) = \mathrm{Specm}(\mathfrak{Z}(A))/\Gamma$ *(the set of* S_n*-orbits)* $= (\mathfrak{h}^*)^{\oplus n}/(S_n \wr W, \bullet)$, *where* $\bullet$ *is the twisted action, and* W *the Weyl group of* $\mathfrak{g}$.

(3) *The set* $CC(\chi)$ *of weights* $\lambda \in H\widehat{\ }$, *corresponding to a given central character* χ, *is nonempty and finite.*

In part (1),
- the isomorphism is by Chevalley's Theorem.

In part (2),
- the first equality is a special case of the Nagata-Mumford Theorem (reductive groups acting on affine varieties);

- the second equality is from Harish-Chandra's theorem and the following "compatibility" of W and S_n on $H^\frown = (\mathfrak{h}^*)^{\oplus n}$:

$$(\sigma \mathbf{w}) \bullet \lambda = (\sigma(\mathbf{w})\sigma) \bullet \lambda = \sigma(\mathbf{w} \bullet \lambda) = \sigma(\mathbf{w}) \bullet \sigma(\lambda)$$

Part (3)

- also holds if $A = A_q = (U_q(\mathfrak{g}))^{\otimes n}$ or $A_\kappa = \mathcal{H}_\kappa^{\otimes n}$, since both have "large enough" centers (e.g. Jantzen, Tikaradze).
- allows for a nice "central character decomposition" of various classes of modules.
- does not hold for $A = A_{q,\kappa}$ for $\kappa \neq 0$, since the center is trivial. What does one do then?

0.7. **The BGG Category.**

Category $\mathcal{O}$ is the full subcategory of all finitely generated $A \rtimes \Gamma$-modules, on which H acts semisimply with finite-dimensional weight spaces, and N_+ acts locally finitely.

- Say $\Gamma = 1$. Then in $\mathcal{O}_A$, for each weight $\lambda \in H\widehat{\ }$, $M_A(\lambda) =$ *Verma module*, $L_A(\lambda) =$ its unique simple quotient.
- For general Γ as well, one has $\{M(x), L(x) : x \in X\} \subset \mathcal{O}$.
- $\mathcal{O}$ is closed under quotienting but not extensions, and every object has a finite filtration with factors the quotients of Verma modules $M(x)$.

Theorem 3.

(1) *$\mathcal{O}$ is finite length if and only if each $M(x)$ is finite length.*

(2) *Equation (1) holds if we replace V by any $L(x)$, or any $M(x)$ - then also replace $L_A(\sigma(\lambda))$ by $M_A(\sigma(\lambda))$.*

(3) *Complete reducibility holds in $\mathcal{O} = \mathcal{O}_{A \rtimes \Gamma}$ if and only if it holds in $\mathcal{O}_A$.*

0.8. **Substitute for central characters.**

- To bypass the problem of trivial center, define $S(x)$ as the equivalence closure of the relations: $[M(x) : L(x')] > 0$, and “restricted duality”.
- $A \rtimes \Gamma$ satisfies *Condition (S)* if every $S(x)$ is finite.
- For instance, $S(x) \subset CC(\chi_x)$, so Condition (S) is satisfied for any subgroup of S_n, with $A = \mathfrak{U}(\mathfrak{g}^{\oplus n})$, A_q, A_κ (for $\kappa \neq 0$).
- It is also satisfied for $A_{q,\kappa}$ (for $\kappa \neq 0$).

0.9. **In the general setup.** Another common property of all these algebras: the *PBW Property* and a *Triangular Decomposition*: $A \cong B_- \otimes H \otimes B_+$ as vector spaces, with H a commutative Hopf algebra (so $H\widehat{\ }$ is the *group* of weights for A).

Now suppose $A_1, \dots, A_n$ are such algebras, and a finite group Γ acts on $A := \otimes_i A_i$ and "permutes the simple roots of A". (And some other technical assumptions on H-weight spaces etc.)

Examples: All the variants mentioned above.

Theorem 4.

(1) *Complete reducibility holds inside the respective categories* $\mathcal{O}$, *at once or never, for* A, *or all* A_i*'s, or* $A \rtimes \Gamma$.

(2) $\widehat{H_A} = \times_i \left(\widehat{H_{A_i}}\right)$, *and if* $\lambda = (\lambda_1, \ldots, \lambda_n)$, *then*

  (a) $L_A(\lambda) = \otimes_i L_{A_i}(\lambda_i)$.

  (b) $M_A(\lambda) = \otimes_i M_{A_i}(\lambda_i)$.

  (c) $S_A(\lambda) = \times_i S_{A_i}(\lambda_i)$.

(3) *If Condition (S) holds for any* A', *then it holds for* $A' \rtimes \Gamma$.
It holds for A *if and only if it holds for all* A_i.

(4) *If for any such* A, Γ, *Condition (S) holds for* $A \rtimes \Gamma$, *then* $\mathcal{O} = \bigoplus \mathcal{O}(x)$.

More about the blocks $\mathcal{O}(x)$:

- Each summand $\mathcal{O}(x)$ has enough projectives and is a (finite length, abelian) highest weight category with BGG Reciprocity.
- Morphisms and extensions between different $\mathcal{O}(x)$'s are trivial.
- Each $\mathcal{O}(x)$ is equivalent to modules over a finite-dimensional quasi-hereditary algebra (Cline, Parshall and Scott).