# PROBLEMS ON WEIGHTS AND HOPF ALGEBRAS

APOORVA KHARE

Throughout, fix an arbitrary ground field $k$.

## 1. Weights

**Definition 1.** Given a $k$-algebra $A$, the set of *weights* is $\mathrm{Hom}_{k-alg}(A, k)$. Given a weight $\lambda$ and an $A$-module $M$, the $\lambda$-*weight space* of $M$ is $M_\lambda := \{m \in M : am = \lambda(a)m \ \forall a \in A\}$.

**Exercise 1.** Now compute the weights for a few algebras.

(1) Suppose $\mathfrak{h}$ is an abelian Lie algebra over any field $k$. Find the set of weights of $\mathrm{Sym}\,\mathfrak{h}$. Check that this is a group under addition. (This is the group of weights for *semisimple Lie algebras*.)
(2) Suppose $\Gamma$ is the free abelian group generated by $K_1, \dots, K_n$. Find the weights of its group algebra, namely, the Laurent polynomial algebra $k[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$. Check that this is a group under multiplication. (These will give the group of (highest) weights for the *quantum group*.)
(3) More generally, show that the weights of any group algebra $k\Gamma$ are the set of *(group) characters* = group homomorphisms : $\Gamma \to k^\times$.
(4) If $H_1, H_2$ are $k$-algebras, and $G_1, G_2$ their sets of weights respectively, then the set of weights of $H_1 \otimes H_2$ is precisely $G_1 \times G_2$. (Generalize this to $k$-algebras $H_1, \dots, H_n$.)
(5) Suppose, for example, that we have a finite set of groups $\Gamma_1, \dots, \Gamma_n$; define $H_i := k\Gamma_i$ and $H = \otimes_{i=1}^n H_i$. The question is: does there exist a group $\Gamma$, such that $H = k\Gamma$?

**Exercise 2.** Suppose $H \subset A$ are $k$-algebras, $M$ is an $A$-module, and $Z := \mathfrak{Z}(A)$ is the center of $A$. Prove the following:

(1) The sum of $H$-weight spaces $M_\lambda$ is direct.
(2) If $M$ is an $H$-diagonalisable $A$-module, and $N \subset M$ is an $A$-module, then $N$ is also $H$-diagonalisable. In other words, if $M = \bigoplus_\lambda M_\lambda$, then $N = \bigoplus_\lambda N_\lambda$, where $N_\lambda$ is defined to be $N_\lambda := N \cap M_\lambda \ \forall \lambda$.
(3) Each $z \in Z$ preserves the $H$-weight spaces of $M$.
(4) If $\dim M_\lambda = 1$ and $M$ is generated by $M_\lambda$ for some $H$-weight $\lambda$ (i.e., $M = A \cdot M_\lambda$), then $Z$ acts (on $M_\lambda$, and hence) on all of $M$ by a *character* (i.e., an algebra map) $\chi_\lambda : Z \to k$. (We will use this later, to study *Verma modules*.) This is done in two parts:
   - Each $z \in Z$ acts by a scalar on all of $M$. (This gives a linear map $\chi_\lambda : Z \to k$.)

- This map $\chi_\lambda$ is not just linear, it is an algebra map.

## 2. Algebras and coalgebras

Now you have to read the definitions of algebras and coalgebras (see Wikipedia or PlanetMath). Then we have:

**Definition 2.**

(1) Henceforth, define $G := \mathrm{Hom}_{k-alg}(H, k)$ to be the set of weights of $H$.
(2) In a coalgebra $(H, \Delta, \varepsilon)$, an element $h$ is said to be *grouplike* if $\Delta(h) = h \otimes h$. Similarly, $h \in H$ is *primitive* if $\Delta(h) = h \otimes 1 + 1 \otimes h$ - and more generally, $h \in H$ is *skew-primitive* if $\Delta(h) = g \otimes h + h \otimes g'$, for $g, g'$ grouplike in $H$.
(3) Denote the set of grouplike elements of $H$ by $G(H)$ - this is *different* from the set of weights $G$ (which will be denoted by $G_H$, if the dependence on $H$ has to be made clear).
(4) We say that an algebra $H$ is *commutative* if $m \circ \tau = m$ (where $\tau : H \otimes H \to H \otimes H$ is the *flip map*, so that the above condition is: $a \cdot b = m(a \otimes b) = m(b \otimes a) = b \cdot a$).

**Exercise 3.** Suppose $(H, m, \eta)$ is a finite-dimensional $k$-algebra. Verify that $(H^*, m^*, \eta^*)$ is a finite-dimensional $k$-coalgebra, as follows: $m^*(h^*)$ should be an element of $H^* \otimes H^* \cong (H \otimes H)^*$ ($\cong$ when $\dim_k H < \infty$). Which one, can only be verified by evaluating against **all** elements of $H \otimes H$. Similarly, how does $\eta^*$ go from $H^*$ to $k$? The claim is that

$$m^*(h^*)(h \otimes h') := h^*(h \cdot h') = h^*(m(h \otimes h')), \qquad \eta^*(h^*) := h^*(1)$$

give us a comultiplication and a counit that make up a coalgebra structure on $H^*$.

**Exercise 4.** Prove that for $H$ as above (i.e., a finite-dimensional $k$-algebra), the set of weights $G$ is precisely the set of grouplike elements in $H^*$!

Moreover, if $H$ is commutative, then $H^*$ is cocommutative.

**Exercise 5.** Similarly, suppose $(H, \Delta, \varepsilon)$ is a finite-dimensional $k$-coalgebra.

(1) Check that the set of grouplike elements in $H$ is linearly independent.
(2) Verify that $(H^*, \Delta^*, \varepsilon^*)$ is a finite-dimensional $k$-algebra. (Thus - first define the maps $\Delta^*, \varepsilon^*$ for $H^* \otimes H^*, H^*$ respectively; and then prove the result.)
(3) And - if $H$ is cocommutative, then $H^*$ is commutative.
(4) If $h \in H$ is grouplike, then $\varepsilon(h) = 1$.
(5) If $h \in H$ is primitive - or more generally, skew-primitive, then $\varepsilon(h) = 0$.
(6) Check that $\mathrm{Sym}\,\mathfrak{h}$ is cocommutative, as is $k\Gamma$, the group algebra of any group $\Gamma$.

Now comes one of the most important concepts - that of *convolution* of two linear maps. This translates to

- addition in $\mathfrak{h}^*$ - in Lie algebras; and
- multiplication of the weights of $k[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$.

**Definition 3.** $H$ is any coalgebra. Given *any linear maps* $\lambda, \mu \in H^*$, define their *convolution* to be $\lambda * \mu \in H^*$, defined by:

$$(\lambda * \mu)(h) := (\lambda \otimes \mu)(\Delta(h)).$$

**Exercise 6.** Check the following:

1. $(\lambda * \mu)(h) = \lambda(h)\mu(h)$ if $h$ is grouplike. (Check it for $k[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$.)
2. $(\lambda * \mu)(h) = \lambda(h) + \mu(h)$ if $h$ is primitive. (Check this for Lie algebras $\mathfrak{h} \subset \operatorname{Sym} \mathfrak{h}$.)
3. In general, convolution is associative.

## 3. Bialgebras and Hopf algebras

**Definition 4.** A *k-bialgebra* is simultaneously a $k$-algebra $(H, m, \eta)$ and coalgebra $(H, \Delta, \varepsilon)$, such that $\Delta$ and $\varepsilon$ are algebra maps.

**Exercise 7.** Suppose $H$ is now a finite-dimensional $k$-bialgebra.

1. Then so is $H^*$ (define all operators!).
2. The set of grouplike elements in $H$ is now an associative monoid. (Applying this to $H^*$, and using Exercise 4, the set of weights $G$ is now a monoid, with unit $\varepsilon$!)
3. Check that for any group $\Gamma$, the group algebra $k\Gamma$ is a bialgebra. So is $\mathfrak{U}\mathfrak{g}$ for any Lie algebra - where $\mathfrak{g}$ is primitive, and we extend the product to words in $\mathfrak{g}$ by multiplicativity.
4. The set of primitive elements in $H$ forms a Lie subalgebra under the usual commutator bracket: $[h, h'] := hh' - h'h$.

Now read the definition of a *Hopf algebra*.

**Exercise 8.** Let's find the antipodes of certain elements. Suppose $H$ is a $k$-Hopf algebra.

1. Show that if $h \in H$ is grouplike, then $S(h) = h^{-1}$. (So $h$ has an inverse.)
2. Show that the set of grouplike elements in $H$ is indeed a group.
3. Show that if $h \in H$ is primitive, then $S(h) = -h$.
4. Suppose $h \in H$ is skew-primitive: $\Delta(h) = g_1 \otimes h + h \otimes g_2$. We already saw that $\varepsilon(h) = 0$; now compute $S(h)$.
5. Show that the group of weights $G$ of $H$ is a group under convolution - as we saw, it already was a monoid - under the inverse map: $\lambda^{-1}(h) := \lambda(S(h))$.

And once again - verify *all* of this for the two standard examples of cocommutative (but not necessarily commutative) Hopf algebras:

**Exercise 9.** Suppose we are given any group $\Gamma$ and any Lie algebra $\mathfrak{g}$ over any ground field $k$.

(1) Verify that $k\Gamma$ and $\mathfrak{U}\mathfrak{g}$ are cocommutative Hopf algebras.
(2) Show that the set of grouplike elements in them are $\Gamma, 1$ respectively, and the set of primitive elements are $0, \mathfrak{g}$ respectively.
(3) Verify that $\mathfrak{U}\mathfrak{g} \otimes k\Gamma$ is also a Hopf algebra.
(4) In general, given two Hopf algebras, how do you define a Hopf algebra structure on their "commuting" tensor product $H_1 \otimes H_2$? ("Commuting" means that multiplication is component-wise. Now define all other maps and check all axioms.) What would the group of weights be here?

**Some big theorems.** In fact, the second part of this exercise is not atypical, as the first of a few big theorems tells us. In all of them, fix $k = \mathbb{C}$.

- **Theorem.** (Cartier-Kostant-Milnor-Moore) *Every* cocommutative Hopf algebra is generated by its primitive and grouplike elements.
- **Theorem.** (Zhu) For a given prime $p \in \mathbb{N}$, *every* Hopf algebra of dimension $p$ is isomorphic to a group algebra (hence, of $\mathbb{Z}/p\mathbb{Z}$).
- **Theorem.** (Etingof-Gelaki) Given any two primes $p, q \in \mathbb{Z}$ (not necessarily distinct), and a $\mathbb{C}$-Hopf algebra $H$ that is *semisimple* (as an algebra),
  - $H$ is *Frobenius* - i.e., the dimension of every irreducible $H$-module, divides $\dim H$; and
  - $H$ is *trivial* - i.e., one of $H$ and $H^*$ is a group algebra.

Now for some results relating $H$ and $H^*$:

**Exercise 10.** Suppose $H$ is a finite-dimensional Hopf algebra over a ground field $k$.

(1) Then so is $H^*$ (i.e., given an earlier problem, check that there exists an antipode for $H^*$, and that it satisfies the antipode axiom in $H^*$).
(2) The set of weights $G$ for $H$, is a commutative group if and only if $H$ is cocommutative.
(3) Now note that $G$ is *always* cocommutative for any such $H$, since $G$ is a group! But $H$ was never assumed to be commutative. What fact are we missing here?

*Proof.* Here is a solution to part (2):

$H$ is cocommutative
$\Leftrightarrow \Delta^{op}(h) = \Delta(h)\ \forall h \in H$
$\Leftrightarrow\ \forall(\lambda, \mu) \in G_{H\otimes H} = G_H \times G_H, (\lambda, \mu)(\Delta^{op}(h)) = (\lambda, \mu)(\Delta(h))\ \forall h \in H$
$\Leftrightarrow\ \forall \lambda, \mu \in G = G_H, (\mu \otimes \lambda)(\Delta(h)) = (\lambda \otimes \mu)(\Delta(h))\ \forall h \in H$
$\Leftrightarrow\ \forall \lambda, \mu \in G = G_H, \mu * \lambda = \lambda * \mu$
$\Leftrightarrow G_H$ is commutative. □

**Fact.** $S$ is an algebra and a coalgebra antihomomorphism. (Formulate this in terms of $m, \Delta, \tau$, where $\tau$ is the flip map!)

**Exercise 11.** Prove the following, for an element of any Hopf algebra $H$:

(1) $\sum S(h)_{(1)} \otimes S(h)_{(2)} = \sum S(h_{(2)}) \otimes S(h_{(1)})$. Here, the subscripts denote *Sweedler notation.*
(2) For all weights $\mu$ of $H$, $\mu(h - S^2(h)) = 0$.

## 4. The origins of Hopf algebras - representations!

Representations (modules) of arbitrary associative algebras have very nice properties: they are vector spaces, they have submodules, quotients, direct sums, and so on. However, one property that works for vector spaces does not go through in general for modules: the *tensor product* of two $A$-modules is not an $A$-module for all $A$.

How does one go about correcting this? Given $A$-modules $M, N$, the vector space $M \otimes N$ is naturally an $A \otimes A$-module, via: $(a \otimes b) \cdot (m \otimes n) := (a \cdot m) \otimes (b \cdot n)$. So if we want a natural map, then we would like a vector space map $\Delta : A \to A \otimes A$, such that any representation $\rho_{M \otimes N} : A \to \mathrm{End}_k(M \otimes N)$ factors through $\rho_M \otimes \rho_N : A \otimes A \to \mathrm{End}_k(M) \otimes \mathrm{End}_k(N)$.

(In short, the *category of representations* should have nice properties, and these requirements motivated Hopf algebras: see the wikipedia entry on "Representation of a Hopf algebra": we want the category to be *strictly monoidal with respect to the tensor product* - and this gives bialgebras, as seen now.)

- This means that if we want things to be as *natural* as possible, $\Delta$ should be an algebra map.
- There is one more requirement: the natural isomorphism of flipping : $(M \otimes N) \otimes K \to M \otimes (N \otimes K)$ of three modules, should be an $A$-module isomorphism.
- There is one more nice property that might be desirable: the one-dimensional vector space $k$ is an $A$-module, in a way that $k \otimes M \cong M \cong M \otimes k$ as $A$-modules.

Guess what all these requirements and nice properties translate into? The answer is - another exercise!

**Exercise 12.** Suppose $A$ is any $k$-algebra (so $A$ has a unit), and $\Delta : A \to A \otimes A$ a *linear* map.

(1) Check that $a$ acts on an element in $(M \otimes N) \otimes K$ by $(\Delta \otimes 1)(\Delta(a)) \in A \otimes A \otimes A$.
(2) The condition that $(M \otimes N) \otimes K \to M \otimes (N \otimes K)$ as $A$-modules *for all* $M, N, K$ - is equivalent to the condition that $\Delta$ is coassociative.
    (One way is easy to prove; the other is hard - use $M = N = K = A$ under left multiplication, and compute in two ways, how every $a$ acts on the vector $(1 \otimes 1) \otimes 1 = 1 \otimes (1 \otimes 1)$ inside the *unital* algebra $A \otimes A \otimes A$.)
(3) Now $M \otimes N$ is an $A$-module if $\Delta : A \to A \otimes A$ is an algebra map. (This is how bialgebras come into the picture!)
(4) A one-dimensional representation of $A$ is the same as a $k$-algebra map (or *weight*!) $\varepsilon : A \to k$.

(5) If we want $k \otimes M \cong M \cong M \otimes k$ for all $A$-modules $k$ under the natural isomorphism $1 \otimes m \leftrightarrow m \leftrightarrow m \otimes 1$), then this requires that $\varepsilon$ satisfies the counit axiom. (Once again, one way is easy, and for the other, use $M = A$ under left multiplication.)

(6) Thus, all these desirable properties hold for $A$-modules (closed under tensoring; tensoring is associative; tensoring with a special one-dimensional module changes nothing) if the associative $k$-algebra $A$ is actually a bialgebra!

(7) One last, fun one: the flip map : $M \otimes N \to N \otimes M$ is a vector space isomorphism, of course, but it is an isomorphism *as modules* if and only if the bialgebra $H$ is cocommutative.

Actually, if $M \otimes N, N \otimes M$ are not isomorphic in general, then we need a *braiding*, or an *R-matrix*. These terms show up in the context of quantum groups, because these are not cocommutative.

Now for some duality, and verifications!

**Exercise 13.** Suppose $H$ is any $k$-algebra.

(1) The category of $H$-modules is closed under taking (linear) duals, if there exists an *algebra anti-homomorphism* $S : H \to H$.

(2) (Henceforth, $H$ is a Hopf algebra.) For $H = k\Gamma$, check that the tensor product, trivial representation (which tensors with anything to give that thing again), and dual representation are defined using precisely the comultiplication, counit, and antipode maps respectively!

Thus, that groups are Hopf algebras, is reflected not only in their structure axioms, but also in their module categories!

(3) Repeat this verification for $H = \mathfrak{U}\mathfrak{g}$ for any (not necessarily commutative) Lie algebra $\mathfrak{g}$.

## 5. The adjoint action and weights

**Definition 5.** $H$ is a Hopf algebra.

(1) The *group adjoint action* of a grouplike $h \in H$ on $H$ is: $Ad(h)(h') = hh'h^{-1}$.

(2) The *Lie algebra adjoint action* of a primitive $h \in H$ on $H$ is: $\operatorname{ad} h(h') = [h, h'] = hh' - h'h$.

(3) The *Hopf algebra adjoint action*, in general, is:

$$\operatorname{ad} h(h') := \sum h_{(1)} h' S(h_{(2)}).$$

**Exercise 14.** Show that if $h$ is grouplike (resp. primitive), then the Hopf algebra adjoint action is the usual group or Lie algebra adjoint action respectively.

Now we get to some applications, involving weights.

**Exercise 15.** Suppose $A$ is an algebra containing $H$. Prove that $A$ is an $\operatorname{ad} H$-module, i.e., $\operatorname{ad} : H \to \operatorname{End}_k(A)$ is an algebra map!

Here's another action; this time, the group of weights $G$ of $H$, acts on $H$ itself!

**Exercise 16.** Define $\rho : G \to \operatorname{End}_k(H)$ via: $\rho(\mu) : h \mapsto ((\mu \circ S) \otimes id_H)(\Delta(h))$.

Then (a) $\rho(\mu)$ is an algebra map : $H \to H$, and (b) $\rho$ is a group homomorphism : $G \to Aut_{k-alg}(H)$.

(For instance, $\rho_\mu(h) = h - \mu(h)$ for primitive $h$, and $\rho_\mu(h) = \mu(h)^{-1}h$ for grouplike $h$. For a *real* example: a linear change of variables : $k[X,Y] \to k[X,Y]$, given by $X \mapsto X+1,\ Y \mapsto Y-3$.)

Finally, suppose again that $H \subset A$ are $k$-algebras, with $H$ a Hopf algebra. For each $\mu \in G$, define $A_\mu$, the $\operatorname{ad} H$-weight space in $A$ of weight $\mu$. (Thus, $\operatorname{ad} h(a_\mu) = \mu(h)a_\mu\ \forall h \in H$.)

For example, if $H = \operatorname{Sym}\mathfrak{h}$, then $A_\mu = \{a \in A : [h,a] = \mu(h)a\}$; similarly, since $KeK^{-1} = q^2 e$ in the *quantum group* $U_q(\mathfrak{sl}_2)$, hence $e$ is in the $q^2$-weight space. Here is another calculation that we generalize: if $m_\lambda \in M_\lambda$ is a weight vector in an $\mathfrak{h}$-module $M$, then $h \cdot a_\mu m_\lambda$ equals:

$$= (ha_\mu - a_\mu h)m_\lambda + a_\mu(hm_\lambda) = [h, a_\mu]m_\lambda + a_\mu\lambda(h)m_\lambda = (\mu(h)+\lambda(h))a_\mu m_\lambda,$$

so that $A_\mu M_\lambda \subset M_{\mu+\lambda}$. Now formulate a similar statement for a group algebra in place of $\operatorname{Sym}\mathfrak{h}$.

**Exercise 17.**

(1) Suppose $H = \operatorname{Sym}\mathfrak{h}$ for an abelian Lie algebra $\mathfrak{h}$. Then $A_\mu \cdot A_\lambda \subset A_{\mu+\lambda}$. Similarly, consider an $A$-module $M$ and its weight space $M_\lambda$. Then $A_\mu M_\lambda \subset M_{\mu+\lambda}$. (Note that one has to verify the weight space on the *entire* $\operatorname{Sym}\mathfrak{h}$, not just on $\mathfrak{h}$ as was done above.)

(2) Similarly, if $H = k\Gamma$ for some group $\Gamma$, then $A_\mu M_\lambda \subset M_{\mu\cdot\lambda}$ - here, $\mu, \lambda$ are weights - or characters - of $\Gamma$.

(3) *Most generally* - the statement is as follows: $A_\mu M_\lambda \subset M_{\mu*\lambda}$ for weights $\mu, \lambda$ of any Hopf algebra $H \subset A$, and weight spaces $A_\mu$ for $\operatorname{ad} H$, and $M_\lambda$ inside an $A$-module $M$.

(4) A much easier variant that you might want to do *before* the previous part: suppose we are given $H \subset A$ as above, and two $H$-*diagonalisable* $A$-modules $M, N$. Then $(M \oplus N)_\lambda = M_\lambda \oplus N_\lambda$ (this holds for any $k$-algebra $A$), and $M \otimes N$ is also semisimple here. More precisely, fill in what $(-)$ should be, and prove the following!

$$(M \otimes N)_\lambda = \bigoplus_{\mu \in G} M_\mu \otimes N_{(-)},$$

where $(G, *)$ is the group of weights.

## 6. (Co)commutativity

Here is a

**Fact.** If $H$ is commutative or cocommutative, then $S^2 = id$.

Now we get to some applications to $A \supset H$.

**Exercise 18.** (Problem due to Joseph.) Suppose $H$ is any Hopf algebra.

(1) Prove that the center of $H$ is its $\operatorname{ad} H$-weight space $H_\varepsilon$. (First verify this for both standard examples of $H$ above.)
(2) Now generalize this to when $H$ is any Hopf algebra inside a $k$-algebra $A$ - to find the set of elements in $A$ that commute with all of $H$.