

# THE SUM OF A FINITE GROUP OF WEIGHTS OF A HOPF ALGEBRA

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ABSTRACT. We evaluate the sum of a finite group of linear characters of a Hopf algebra, at all grouplike and skew-primitive elements, and mention results for products of skew-primitive elements. Examples include groups, (quantum groups over) Lie algebras, the small quantum groups of Lusztig, and their variations (by Andruskiewitsch and Schneider).

## 1. INTRODUCTION

1.1. **Motivation.** Suppose  $G$  is a finite group, and we consider the irreducible characters  $\widehat{G} = \{\chi : \mathbb{C}G \rightarrow \mathbb{C}\}$ , over  $\mathbb{C}$ . Then we have the *orthogonality relations* for characters. They imply the following: if we define  $\theta_\chi := \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1})g \in H = \mathbb{C}G$ , then  $\nu(\theta_\chi) = 0$  for  $\chi \neq \nu \in \widehat{G}$ .

Similarly, the other orthogonality relation (for columns) implies that  $\sum_{\chi \in \widehat{G}} (\dim \rho_\chi) \chi(g)$  is either zero or a factor of  $|G|$ , depending on whether or not  $g = 1$ . (Here,  $\rho_\chi$  is the irreducible representation with character  $\chi$ .)

In what follows, we work over a commutative integral domain  $R$ . Note that there is an analogue of the first equation above for any  $R$ -algebra  $H$  that is a free  $R$ -module - for linear characters, or *weights*: given an algebra map  $\lambda : H \rightarrow R$ , we define a *left  $\lambda$ -integral* of  $H$  to be any (nonzero)  $\Lambda_L^\lambda \in H$  so that  $h \cdot \Lambda_L^\lambda = \lambda(h)\Lambda_L^\lambda$  for all  $h \in H$ . One can similarly define right and two-sided  $\lambda$ -integrals in  $H$ . (For instance, the  $\theta_\chi$ 's above, are two-sided  $\chi$ -integrals for any weight  $\chi$ .) Then we have the following result:

**Lemma 1.1.** *If  $\lambda \neq \nu$  are weights of  $H$  with corresponding nonzero left integrals  $\Lambda_L^\lambda, \Lambda_L^\nu$  respectively, then  $\nu(\Lambda_L^\lambda) = 0 = \Lambda_L^\lambda \Lambda_L^\nu$ .*

*Proof.* Choose  $h$  so that  $\lambda(h) \neq \nu(h)$ . Then

$$\nu(h)\nu(\Lambda_L^\lambda)\Lambda_L^\nu = h\Lambda_L^\lambda \cdot \Lambda_L^\nu = (h \cdot \Lambda_L^\lambda) \cdot \Lambda_L^\nu = \lambda(h)\Lambda_L^\lambda \cdot \Lambda_L^\nu$$

whence  $\nu(\Lambda_L^\lambda) \cdot (\lambda(h) - \nu(h))\Lambda_L^\nu = 0$ . Since we are working over an integral domain and within a free module, hence this implies that  $\nu(\Lambda_L^\lambda) = 0$ . Moreover,  $\Lambda_L^\lambda \Lambda_L^\nu = \nu(\Lambda_L^\lambda)\Lambda_L^\nu = 0$ .  $\square$

Thus, we now seek a ‘‘Hopf-theoretic’’ analogue for the second orthogonality relation (which might involve only weights, and not all irreducible

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characters). Note that in general, a Hopf algebra might not have a nontrivial (sub)group of linear characters that is finite (for instance,  $\mathcal{U}\mathfrak{g}$  for a complex Lie algebra  $\mathfrak{g}$ ). However, if such subgroups do exist, then we try below, to evaluate the sum (i.e., the coefficient of each weight is  $\dim \rho = 1$ ) of all the weights in them, at various elements of  $H$ . For example, one may ask: does this sum always vanish at a nontrivial grouplike  $g \in H$ ?

**1.2. One of the setups, and some references.** Instead of attempting a summary of the results, which are several and computational, we make some remarks. Let  $\Pi$  be a finite group of weights of a Hopf algebra  $H$  over a commutative unital integral domain  $R$ . One has the notion of *grouplike* elements (i.e.,  $\Delta(g) = g \otimes g$ ) and *skew-primitive* elements in  $H$ .

We define  $\Sigma_{\Pi} := \sum_{\gamma \in \Pi} \gamma : H \rightarrow R$ , and compute  $\Sigma_{\Pi}$  at all grouplike and skew-primitive elements in  $H$ . In a wide variety of examples - including that in [3] - the computations reduce to grouplike elements. In other words, there are several families of algebras, where knowing  $\Sigma_{\Pi}$  at grouplike elements effectively tells us  $\Sigma_{\Pi}$  at *all* elements.

We then attempt to evaluate  $\Sigma_{\Pi}$  at products of skew-primitive elements. Once again, in the spirit of the previous paragraph, there are numerous examples of Hopf algebras generated by grouplike and skew-primitive elements in the literature. The first two examples below are from folklore, and references can be found in [13].

- (1) By the Cartier-Kostant-Milnor-Moore Theorem (e.g., see [13, Theorem 5.6.5]), every cocommutative connected Hopf algebra  $H$  over a field of characteristic zero, is of the form  $\mathcal{U}\mathfrak{g}$ , where  $\mathfrak{g}$  is the set of primitive elements in  $H$ . Similarly, every complex cocommutative Hopf algebra is generated by primitive and grouplike elements.
- (2) If our Hopf algebra is pointed (and over a field), then by the Taft-Wilson Theorem [13, Theorem 5.4.1.1], we have evaluated  $\Sigma_{\Pi}$  on any element of  $C_1$ , the first term in the coradical filtration (which is spanned by grouplike and skew-primitive elements).
- (3) The final example is from a recent paper [3]. The Classification Theorem 0.1 says, in particular, that if  $H$  is a finite-dimensional pointed Hopf algebra over an algebraically closed field of characteristic zero, and the grouplike elements form an abelian group of order coprime to 210, then  $H$  is generated by grouplike and skew-primitive elements, and is a variation of a small quantum group of Lusztig.

Let us conclude with one of our results. We say that an element  $h \in H$  is *pseudo-primitive* with respect to  $\Pi$  if  $\Delta(h) = g \otimes h + h \otimes g'$  for grouplike  $g, g'$  satisfying  $\gamma(g) = \gamma(g')$  for all  $\gamma \in \Pi$ . (We mention the respective references in the paper, after stating the results.)

**Theorem 1.1.** *Fix  $n \in \mathbb{N}$ , as well as the  $R$ -Hopf algebra  $H$  and a finite subgroup of weights  $\Pi$  of  $H$ . Suppose  $h_1, \dots, h_n \in H$  are pseudo-primitive*

with respect to  $\Pi$ , and  $\Delta(h_i) = g_i \otimes h_i + h_i \otimes g'_i$  for all  $i$ . Define  $\mathbf{h} = \prod_i h_i$ , and similarly,  $\mathbf{g}, \mathbf{g}'$ .

- (1) If  $\text{char}(R) = 0$  or  $\text{char}(R) \nmid |\Pi|$ , then  $\Sigma_\Pi(\mathbf{h}) = 0$ .
- (2) Suppose  $0 < p = \text{char}(R)$  divides  $|\Pi|$ , and  $\Pi_p$  is any  $p$ -Sylow subgroup. If  $\Pi_p \not\cong (\mathbb{Z}/p\mathbb{Z})^m$  for any  $m > 0$ , then  $\Sigma_\Pi(\mathbf{h}) = 0$ .
- (3) ( $p$  as above.) Define  $\Phi := \Pi/[\Pi, \Pi]$ , and by above, suppose  $\Phi_p \cong (\mathbb{Z}/p\mathbb{Z})^k$  is a  $p$ -Sylow subgroup of  $\Phi$ . Let  $\Phi'$  be any Hall complement(ary subgroup); thus  $|\Phi'| = |\Phi|/|\Phi_p|$ . Then

$$\Sigma_\Pi(\mathbf{h}) = |[\Pi, \Pi]| \cdot \Sigma_{\Phi'}(\mathbf{g}) \cdot \Sigma_{\Phi_p}(\mathbf{h})$$

- (4) If  $\Sigma_{\Phi_p}(\mathbf{h})$  is nonzero, then  $(p-1)|n$ , and  $0 \leq k \leq n/(p-1)$ . (Moreover, examples exist wherein  $\Sigma_{\Phi_p}(\mathbf{h})$  can take any value  $r \in R$ .)

These results occur below as Proposition 5.4, and Theorems 7.2, 7.1, and 7.3 respectively.

## 2. GROUPLIKE ELEMENTS AND QUANTUM GROUPS

**2.1. Preliminaries.** We first set some notation, and make some definitions.

**Definition 2.1.** Suppose  $R$  is a commutative unital integral domain.

- (1) *Integers* in  $R$  are the image of the group homomorphism  $\mathbb{Z} \rightarrow R$ , sending  $1 \mapsto 1$ .
- (2) A *weight* of an  $R$ -algebra  $H$  is an  $R$ -algebra map  $\nu : H \rightarrow R$ . Denote the set of weights by  $\Gamma$  (or  $\Gamma_H$ ). Given  $\nu \in \Gamma_H$ , the  $\nu$ -*weight space* of an  $H$ -module  $V$  is  $V_\nu := \{v \in V : h \cdot v = \nu(h)v \ \forall h \in H\}$ .
- (3) Given a free  $R$ -module  $H$ , define  $H^* := \text{Hom}_{R\text{-mod}}(H, R)$ .
- (4) We denote an  $R$ -Hopf algebra  $H$  as an  $R$ -algebra  $(H, \mu = \cdot, \eta)$  (where  $\mu, \eta$  are coalgebra maps) as well as an  $R$ -coalgebra  $(H, \Delta, \varepsilon)$  (where  $\Delta, \varepsilon$  are algebra maps), equipped also with an antipode  $S$  (which is an  $R$ -(co)algebra anti-homomorphism).
- (5) In a Hopf algebra, an element  $h$  is *grouplike* if  $\Delta(h) = h \otimes h$ , and *primitive* if  $\Delta(h) = 1 \otimes h + h \otimes 1$ . Define  $G(H)$  (resp.  $H_{\text{prim}}$ ) to be the set of grouplike (resp. primitive) elements in a Hopf algebra  $H$ .

For an introduction to Hopf algebras, see [1]. In particular,  $H^*$  is also an  $R$ -algebra under *convolution*  $\Delta^*$ : given  $\lambda, \nu \in H^*$  and  $h \in H$ , one defines  $\langle \lambda * \nu, h \rangle := \langle \lambda \otimes \nu, \Delta(h) \rangle$ . The set  $\Gamma$  of weights is now a group under  $*$ , with inverse given by  $\langle \lambda^{-1}, h \rangle := \langle \lambda, S(h) \rangle$ . We do not have a comultiplication map on  $H^*$  in general (that is dual to the multiplication in  $H$ ); however, in case we do, the set  $\Gamma$  is the group  $G(H^*)$  of grouplike elements in  $H^*$ .

**Standing Assumption 2.1.** For this article,  $H$  is any Hopf algebra over a commutative unital integral domain  $R$ . Fix a finite subgroup of weights  $\Pi \subset \Gamma = \Gamma_H$ .

In general, given a finite subgroup  $\Pi \subset \Gamma$  for any Hopf algebra  $H$ , the element  $\Sigma_\Pi := \sum_{\gamma \in \Pi} \gamma$  is a functional in  $H^*$ , and if  $R$  (i.e., its quotient field) has characteristic zero, then  $\Sigma_\Pi$  does not kill the scalars  $\eta(R)$ . What, then, is its kernel? How about if  $\Pi$  is cyclic, or all of  $\Gamma$  (this, only if  $H$  is  $R$ -free, and finite-dimensional over the quotient field of  $R$ )?

**Lemma 2.1.**  $\Pi \subset \Gamma = \Gamma_H$  as above.

- (1)  $\Sigma_\Pi(1) = 0$  if and only if  $\text{char}(R)$  divides  $|\Pi|$ .
- (2)  $[H, H] \subset \ker \Sigma_\Pi$ .
- (3)  $\Sigma_\Pi(\text{ad } h(h')) = \varepsilon(h)\Sigma_\Pi(h')$  for all  $h, h' \in H$ . In particular, if  $h \in \ker \varepsilon$ , then  $\text{im}(\text{ad } h) \subset \ker \Sigma_\Pi$ .

*Proof.* The first part is easy, and the other two follow because the statements hold if we replace  $\Sigma_\Pi$  by any  $\gamma \in \Gamma$  (since such  $\gamma$ 's are algebra maps).  $\square$

The goal of this section and the next, is to evaluate  $\Sigma_\Pi$  at all grouplike and skew-primitive elements in  $H$ . For these computations, a key fact to note is that for all  $\lambda \in \Pi$ , we have (in the  $R$ -algebra  $H^*$ ):

$$\lambda * \Sigma_\Pi = \sum_{\nu \in \Pi} \lambda * \nu = \Sigma_\Pi = \cdots = \Sigma_\Pi * \lambda \quad (2.1)$$

**Remark 2.1.** We occasionally compute  $\Sigma_\Pi$  with  $H$  an  $R$ -algebra (that is not a Hopf algebra), where  $\Pi \subset \Gamma_H$  has a group structure on it. As seen in Proposition 2.2 below, there is some underlying Hopf algebra in some cases.

**Definition 2.2.** Suppose we have a subset  $\Theta \subset \Gamma$ , and  $\lambda \in \Pi$ .

- (1) For  $g \in G(H)$ , define  $\Gamma_g := \{\gamma \in \Gamma : \gamma(g) = 1\}$ , and  $\Theta_g := \Gamma_g \cap \Theta$ .
- (2) Define  $G_\Theta(H)$  to be the set (actually, normal subgroup) of grouplike elements  $g \in G(H)$  so that  $\gamma(g) = 1$  for all  $\gamma \in \Theta$ .
- (3) For finite  $\Theta$ , define the functional  $\Sigma_\Theta \in H^*$  to be  $\Sigma_\Theta := \sum_{\gamma \in \Theta} \gamma$ . We set  $\Sigma_\emptyset := 0$ .
- (4)  $n_\lambda := o_\Pi(\lambda) = |\langle \lambda \rangle|$  is defined to be the order of  $\lambda$  in  $\Pi$ .

**Remark 2.2.**

- (1) For instance,  $G_{\{\varepsilon\}}(H) = G(H)$ , and  $G(H) \cap [1 + \text{im}(\text{id} - S^2)] \subset G_\Gamma(H)$  because (e.g., see [1]) every  $\gamma \in \Gamma$  equals  $(\gamma^{-1})^{-1}$ .
- (2) For any  $g \in G(H)$  and  $\Theta \subset \Gamma$ ,  $\Theta_g = \Theta$  if and only if  $g \in G_\Theta(H)$ .
- (3)  $\Theta_g$  is a subgroup if  $\Theta$  is.

**2.2. Grouplike elements.** We first determine how  $\Sigma_\Pi$  acts on grouplike elements, and answer the motivating question above, of finding a Hopf-theoretic analogue of the second orthogonality relations for group characters.

**Proposition 2.1** (“Orthogonality” at grouplike elements). *If  $g \in G(H) \setminus G_\Pi(H)$ , then  $\Sigma_\Pi(g) = 0$ . If  $g \in G_\Pi(H)$ , then  $\Sigma_\Pi(g) = |\Pi|$ .*

*Proof.* Apply equation (2.1) to  $g$ , with  $\lambda \notin \Gamma_g$ .  $\square$

Before we look at various examples, we need some notation.

**Definition 2.3.** As always,  $R$  is a commutative unital integral domain. Let  $l$  be a nonnegative integer.

- (1) Define  $\sqrt{l}$  (resp.  $\sqrt[l]{l}$ ) to be the set of ( $l^{\text{th}}$ ) roots of unity in  $R$ . (Thus,  $\sqrt[0]{l} = R^\times$ ,  $\sqrt[1]{l} = \{1\}$ , and  $\sqrt{l} = \cup_{l>0} \sqrt[l]{l}$ .)
- (2) Given  $q \in R^\times$ , define  $\text{char}(q)$  to be the smallest positive integer  $m$  so that  $q^m = 1$ , and zero if no such  $m$  exists.
- (3) The group  $\mathfrak{G}_{n,l}$  is the abelian group generated by  $\{K_i : 1 \leq i \leq n\}$ , with relations  $K_i K_j = K_j K_i, K_i^l = 1$ .
- (4) The group  $\mathfrak{G}_{n,l}^*$  is defined to be  $(\sqrt{l} \cap \sqrt[l]{l})^n$ .

Thus,  $\mathfrak{G}_{n,l}$  is free iff  $l = 0$ ,  $\mathfrak{G}_{n,l}^* = (\sqrt[l]{l})^n \forall l > 0$ , and  $\mathfrak{G}_{n,0}^* = (\sqrt{l})^n$ .

The first example where we apply the above result, is obviously:

**Example 1** (Group rings). For instance, the above result computes  $\Sigma_\Pi$  on all of  $H$ , if  $H$  is a group ring. We present a specific example:  $G = \mathfrak{G}_{n,l}$  (defined above), for (a fixed)  $n \in \mathbb{N}$  and  $l \geq 0$ . Then  $\Gamma = (\sqrt[l]{l})^n$ , and any finite order element  $\gamma \in \Gamma$  maps each  $K_i$  to a root of unity in  $R$ . Thus,  $\Pi \subset (\sqrt{l})^n \cap \Gamma = \mathfrak{G}_{n,l}^*$ .

Let us now compute  $\Sigma_\Pi(g)$  for some  $g = \prod_{i=1}^n K_i^{n_i}$ , where  $n_i \in \mathbb{Z} \forall i$ . Note that the set  $\{\gamma(K_i) : 1 \leq i \leq n, \gamma \in \Pi\}$  is a finite set of roots of unity; hence the subgroup of  $\sqrt{l} \subset R^\times$  that it generates, is cyclic, say  $\langle \zeta \rangle$ . Thus,  $\gamma(K_i) = \zeta^{l_i(\gamma)}$  for some  $l_i : \Pi \rightarrow \mathbb{Z}$ .

The above result now says that  $\Sigma_\Pi(g) = 0$  if there exists  $\gamma \in \Pi$  so that  $|\langle \zeta \rangle|$  does not divide  $\sum_{i=1}^n n_i l_i(\gamma)$ , and  $|\Pi|$  otherwise. (Of course, we can also apply the above result directly to  $g = \prod_i K_i^{n_i}$ .)

**Remark 2.3.** From the above proposition, finding out if  $g \in G_\Pi(H)$  is an important step. However, since  $\beta(g) = 1 \forall \beta \in [\Pi, \Pi]$ , hence it suffices to compute if  $\lambda(g) = 1$ , with  $\lambda$ 's the lifts of a set of generators of the (finite) abelian group  $\Pi/[\Pi, \Pi]$ . We see more on this in Section 4 below.

**2.3. Application to quantum groups and related examples.** We now mention some more examples where the above result applies - Hopf algebras that quantize semisimple Lie algebras, their Borel subalgebras, and polynomial algebras (i.e., coordinate rings of affine spaces/abelian Lie algebras). There are yet other algebras, which are not Hopf algebras but can be treated similarly, as we shall see below.

To do this, we need some basic results; here is the setup for them. Suppose an  $R$ -algebra  $A$  contains a Hopf subalgebra  $H$ , so that  $A$  is an  $\text{ad } H$ -module. Then we have the following result:

**Lemma 2.2.** *Suppose  $\mu \in \Gamma_A$  (i.e.,  $\mu : A \rightarrow R$  is an  $R$ -algebra map). If  $\nu \neq \varepsilon$  in  $\Gamma_H$ , then  $\mu \equiv 0$  on  $A_\nu$ .*

*Proof.* Given  $\nu \neq \varepsilon$ , there is some  $h \in H$  such that  $\nu(h) \neq \varepsilon(h)$ . Now given  $a_\nu \in A_\nu$ , apply  $\mu$  to the equation:  $\text{ad } h(a_\nu) = \nu(h)a_\nu$ . Simplifying, we get:

$$\varepsilon(h)\mu(a_\nu) = \nu(h)\mu(a_\nu)$$

and since we work in a free  $R$ -module,  $\mu(a_\nu) = 0 \forall \mu \in \Gamma_A, \nu \neq \varepsilon_H$ .  $\square$

Applying this easily yields the following result.

**Proposition 2.2.** *(A, H as above.) Suppose an R-algebra A contains H and a vector subspace V, that is of the form  $V = \bigoplus_{\nu \neq \varepsilon} V_\nu$  (for the ad H-action).*

- (1) *Every  $\mu \in \Gamma$  kills AVA.*
- (2) *If  $A = H + AVA$ , then  $\Gamma_A \subset \Gamma_H$ .*
- (3) *Say  $A = H + AVA$ , and  $\Pi \subset \Gamma_A$  is a finite subgroup of weights of A (from above). If  $a \in A$  satisfies  $a - \sum_{g \in G(H)} a_g g \in AVA$  (where  $a_g \in R \forall g$ ), then  $\Sigma_\Pi(a) = |\Pi| \sum_{g \in G_\Pi(H)} a_g$ .*

**Remark 2.4.** Thus, if  $\Pi \subset \Gamma_A$  is a group (i.e., a subset with a group structure on it), then  $\Sigma_\Pi(AVA) = 0$ . Hence, computing  $\Sigma_\Pi$  at any  $a \in A$  essentially reduces to the case of the Hopf subalgebra  $H$ . When  $H$  is a group algebra, Proposition 2.1 above tells us the answer in this case - assuming that the group operation in  $\Pi$  agrees with the one in  $\Gamma_H$ .

Moreover, even though  $A$  is not a Hopf algebra here, we see that the computations come from Hopf algebra calculations (for  $H$ ).

We now apply the above theory to some examples; note that they are not always Hopf algebras. In each case,  $G$  is of the form  $\mathfrak{G}_{n,l}$  for some  $n, l$ . We also choose a special element  $q \in R^\times$  in each case; then  $\text{char}(q)|l$ .

**Example 2** (“Restricted” quantum groups of semisimple Lie algebras). For this example,  $R = k$  is a field with  $\text{char } k \neq 2$ , with a special element  $q \neq 0, \pm 1$ . Suppose  $\mathfrak{g}$  is a semisimple Lie algebra over  $k$ , together with a fixed Cartan subalgebra and root space decomposition (e.g., using a Chevalley basis, as in [8, Chapter 7]).

One then defines the (Hopf) algebra  $U_q(\mathfrak{g})$  as in [9, §4.2,4.3]. In particular, we only note here that it is generated by  $\{K_j^{\pm 1}, e_j, f_j : 1 \leq j \leq n\}$  (here,  $n$  is the rank of  $\mathfrak{g}$ , and we have the simple roots  $\alpha_i$ ), modulo the relations:

$$K_i e_j K_i^{-1} = q^{(\alpha_i, \alpha_j)} e_j, \quad K_i f_j K_i^{-1} = q^{-(\alpha_i, \alpha_j)} f_j, \quad e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$$

where  $q_i = q^{(\alpha_i, \alpha_i)/2}$  for some bilinear form  $(\cdot, \cdot)$  on  $\mathfrak{h}^*$ . We may also need that  $q^4$  or  $q^6$  is not 1, and maybe  $\text{char } k \neq 3$ . The other relations are that  $K_i K_j = K_j K_i$ ,  $K_i^{\pm 1} K_i^{\mp 1} = 1$ , and the (two) quantum Serre relations. Define  $V := \bigoplus_{j=1}^n (k e_j \oplus k f_j)$ .

We define the “restricted” quantum group now; see [12, Chapter 6]. Given some fixed  $l \geq 0$  so that  $q^l = 1$  (whence  $\text{char}(q)|l$ ), define the associative (not necessarily Hopf) algebra  $u_{q,l}(\mathfrak{g})$  to be the quotient of  $U_q(\mathfrak{g})$  by the relations (for all  $j$ )  $K_j^l = 1$ , and  $e_j^l = f_j^l = 0$  if  $l > 0$ . Note that  $u_{q,l}(\mathfrak{g}) = U_q(\mathfrak{g})$  if

$l = 0$ , and  $u_{q,l}(\mathfrak{g})$  is a Hopf algebra if  $l = 0$  or  $\text{char}(q)$ . Moreover, Proposition 2.2 allows us to compute  $\Sigma_{\Pi}$  for all  $l$ .

For each  $j$ , note that  $e_j, f_j$  are weight vectors (with respect to the adjoint action of the abelian group  $G = \mathfrak{G}_{n,l}$  generated by all  $K_i^{\pm 1}$ ) with weights  $q^{\pm \alpha_j} \neq \varepsilon = q^0$ . Hence Lemma 2.2 implies that  $\mu(e_j) = \mu(f_j) = 0 \forall j, \mu$ . Moreover, given the PBW property for  $A = U_q(\mathfrak{g})$ , we know that  $A = k\mathfrak{G}_{n,l} \oplus (V_- A + A V_+)$ , where  $V_+, V_-$  are the spans of the  $e_j$ 's and  $f_j$ 's respectively. Hence  $\mu \in \Gamma \subset \Gamma_G$  by the result above.

Every  $\mu \in \Gamma_G$  is compatible with the commuting of the  $K_i$ 's, the quantum Serre relations (since  $\mu|_V \equiv 0$ ), and the “ $l^{\text{th}}$  power relations”. The only restriction is the last one left, namely:  $0 = \mu([e_i, f_i])$ , which gives us that  $\mu(K_i) = \mu(K_i^{-1})$  for all  $i, \mu$ . Hence  $\mu(K_i) = \pm 1$ , so that  $\Gamma \cong (\mathbb{Z}/2\mathbb{Z})^n \cap (\sqrt[l]{1})^n$ , which is of size  $2^n$  or 1, depending on whether  $l$  is even or odd. We now compute  $\Sigma_{\Pi}(a)$  using the second part of Proposition 2.2, for any  $a \in A$ .

**Example 3** (Restricted quantum groups of Borel subalgebras). We consider the subalgebra  $A' = u_{q,l}(\mathfrak{b})$  of  $A = u_{q,l}(\mathfrak{g})$ , that is generated by  $\{K_i^{\pm 1}, e_i : 1 \leq i \leq n\}$ ; once again, this algebra quantizes the Borel subalgebra  $\mathfrak{b}$  of  $\mathfrak{g}$  if  $l = 0$  (and is a Hopf algebra if  $l = 0$  or  $\text{char}(q)$ ). Moreover, every  $\mu \in \Gamma_{A'}$  kills each  $e_i$  (where we use  $V_+$  for  $V$ ), and we have  $\Gamma \subset \Gamma_G$ , by Proposition 2.2 above. Moreover, all such maps  $\mu \in \Gamma_G$  are now admissible (i.e., extend to the whole of  $A'$ ), so  $\Gamma = \Gamma_G \cong (\sqrt[l]{1})^n$ .

If we look at a finite subgroup  $\Pi \subset \Gamma$ , then as above, we must have  $\Pi \subset \mathfrak{G}_{n,l}^*$  (note that  $R = k$  here), and furthermore, evaluation of  $\Sigma_{\Pi}$  once again reduces to the grouplike case.

**Example 4** (Taft algebras). Given a primitive  $n$ th root  $q$  of unity, the  $n$ th Taft algebra is

$$T_n := R\langle x, g \rangle / (gx - qxg, g^n - 1, x^n)$$

Once again, every weight must kill  $x$ , and sends  $g$  to some power of  $q$ . Hence the set  $\Gamma$  of weights is cyclic, whence so is  $\Pi$ . It is now easy to show:

**Lemma 2.3.** *Every weight kills  $x$ , and  $\Sigma_{\Pi}(g^k) = |\Pi|$  if  $|\Pi| \mid k$ , or 0 otherwise.*

**Example 5** (Quantization of affine space). We refer to [7]; once again,  $R$  is a unital commutative integral domain. The quantum affine space over  $R$  (with a fixed element  $q \in R^\times$ ) is the quadratic algebra  $T_R(V)/(x_j x_i - q x_i x_j, i < j)$ , where  $V := \bigoplus_i R x_i$ . This does not have a Hopf algebra structure; however, Hu does present a quantization of  $R[x] := \text{Sym}_R(V)$  in [7, §5] - that is, the quantum group associated to a “finite-dimensional” abelian Lie algebra. We consider a more general associative (not necessarily Hopf)  $R$ -algebra  $\mathcal{A}_{q,l}(n)$  generated by  $\{K_i^{\pm 1}, x_i : 1 \leq i \leq n\}$ , with the relations:

$$\begin{aligned} K_i K_j &= K_j K_i, & K_i^l &= 1, & K_i^{\pm 1} K_i^{\mp 1} &= 1, \\ K_i x_j K_i^{-1} &= \theta_{ij} q^{\delta_{ij}} x_j, & x_i x_j &= \theta_{ij} x_j x_i \end{aligned}$$

where  $\text{char}(q)|l$ , and  $\theta_{ij}$  equals  $q$  (respectively  $1, q^{-1}$ ) if  $i > j$  (respectively  $i = j, i < j$ ). Note that  $\mathcal{A}_{q, \text{char}(q)}(n) = \mathcal{A}_q(n)$ , the Hopf algebra introduced and studied by Hu, and  $\mathcal{A}_{q,l}(n)$  becomes the Hopf algebra  $R[x]$  if  $q = l = 1$ .

We consider the “nontrivial” case  $q \neq 1$  (we consider the  $q = 1$  case later below). Once again, each  $x_j$  is a weight vector with respect to the (free abelian) group  $G = \mathfrak{G}_{n,l}$ , and no  $x_j$  is in the  $\varepsilon$ -weight space, so every  $\mu \in \Gamma$  kills  $x_j$  for all  $j$ . As in the previous example,  $\Gamma \cong (\sqrt[l]{1})^n$ , and any finite subgroup  $\Pi$  must be contained in  $\mathfrak{G}_{n,l}^*$ .

Moreover, evaluation of  $\Sigma_\Pi$  once again reduces to the grouplike case.

**Example 6** (Quantization of the Virasoro algebra). For this example, we assume that  $R$  is a field, and  $q \in R^\times$  is not a root of unity. We refer to [6, Page 100] for the definitions; the Hopf algebra in question is the  $R$ -algebra  $\mathcal{U}_q$  generated by  $\mathcal{T}, \mathcal{T}^{-1}, c, e_m (m \in \mathbb{Z})$  with relations:

$$\begin{aligned} \mathcal{T}\mathcal{T}^{-1} &= \mathcal{T}^{-1}\mathcal{T} = 1, \\ q^{2m}\mathcal{T}^m c &= c\mathcal{T}^m, \\ \mathcal{T}^m e_n &= q^{-2(n+1)m} e_n \mathcal{T}^m, \\ q^{2m} e_m c &= c e_m \\ q^{m-n} e_m e_n - q^{n-m} e_n e_m &= [m-n] e_{m+n} + \delta_{m+n,0} \frac{[m-1][m][m+1]}{[2][3]\langle m \rangle} c \end{aligned}$$

where  $[m] := \frac{q^m - q^{-m}}{q - q^{-1}}$  and  $\langle m \rangle := q^m + q^{-m}$  for all  $m \in \mathbb{Z}$ .

We can now compute the group of weights, as well as  $\Sigma_\Pi(h)$  for any monomial word  $h$  in the above alphabet. The following is proved using the defining relations above.

**Proposition 2.3.** *Setup as above.*

- (1) *The group of weights is  $\Gamma = (R^\times, \cdot)$  (so every finite subgroup  $\Pi$  is cyclic). A weight  $r \in R^\times$  kills  $c$  and all  $e_n$ , and sends  $\mathcal{T}$  to  $r$ .*
- (2)  *$\Sigma_\Pi(h) = 0$  if the monomial word  $h$  contains  $c$  or any  $e_n$ . Moreover,  $\Sigma_\Pi(\mathcal{T}^m) = 0$  unless  $|\Pi|$  divides  $m$ , in which case  $\Sigma_\Pi(\mathcal{T}^m) = |\Pi|$ .*

**Example 7** (Quantum linear groups). For the definitions, we refer to [5, §2]. The *quantum general* (resp. *special*) *linear group*  $GL_q(n) = R_q[GL_n]$  (resp.  $SL_q(n) = R_q[SL_n]$ ) is the localization of the algebra  $\mathcal{B}$  (defined presently) at the central *quantum determinant*

$$\det_q := \sum_{\pi \in S_n} (-q)^{l(\pi)} \prod_{i=1}^n u_{i, \pi(i)}$$

(resp. the quotient of  $\mathcal{B}$  by the relation  $\det_q = 1$ ). Here, the algebra  $\mathcal{B} = R_q[\mathfrak{gl}(n)]$  is generated by  $\{u_{ij} : 1 \leq i, j \leq n\}$ , with relations

$$\begin{aligned} u_{ik} u_{il} &= q u_{il} u_{ik}, & u_{ik} u_{jk} &= q u_{jk} u_{ik} \\ u_{il} u_{jk} &= u_{jk} u_{il}, & u_{ik} u_{jl} - u_{jl} u_{ik} &= (q - q^{-1}) u_{il} u_{jk} \end{aligned}$$

where  $q \in R^\times$ , and  $i < j, k < l$ .

As above, we work with the “nontrivial” case  $q \neq \pm 1$ . Let us actually compute  $\Gamma$  in both cases. In either case, note that  $\det_q \neq 0$ . Given any permutation  $\pi \in S_n$ , suppose we have  $i < j$  so that  $\pi(i) > \pi(j)$ . Then

$$(q - q^{-1})\mu(u_{i,\pi(i)}u_{j,\pi(j)}) = \mu([u_{i,\pi(j)}, u_{j,\pi(i)}]) = 0$$

whence  $\mu(u_{i,\pi(i)}u_{j,\pi(j)}) = 0$  for all  $\mu \in \Gamma$ .

The only permutations for which this does not happen is  $\{\pi \in S_n : i < j \Rightarrow \pi(i) < \pi(j)\} = \{\text{id}\}$ . Hence  $\mu(\det_q) = \prod_i \mu(u_{ii}) \neq 0$ , whence no  $u_{ii}$  is killed by any  $\mu$ . But now, for  $i < l$ , we have

$$\mu(u_{ii}u_{il}) = q\mu(u_{il}u_{ii}), \quad \mu(u_{ii}u_{li}) = q\mu(u_{li}u_{ii})$$

whence  $\mu(u_{ij}) = 0$  for all  $i \neq j, \mu \in \Gamma$ . In particular, since  $\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}$  for (all  $i, j$  and) both  $GL_q(n)$  and  $SL_q(n)$ , hence  $\Gamma_{GL} \cong (R^\times)^n$ , and  $\Gamma_{SL} \cong (R^\times)^{n-1}$  (both under coordinate-wise multiplication), since  $\mu(u_{nn}) = \prod_{i=1}^{n-1} \mu(u_{ii})^{-1}$  in the latter case.

Finally, computing  $\Sigma_\Pi$  now reduces to the above results and the first example (of the free group  $G = \mathfrak{G}_{n,0}$ ). This is because we reduce any  $h \in GL_q(n)$  or  $SL_q(n)$  to a sum of monomial words, and such a word is not killed by any  $\mu$  if and only if there is no contribution from any  $u_{il}, i \neq l$ .

**Example 8** (Hopf regular triangular algebras). These were defined in [10]; we mention the definition first.

**Definition 2.4.** An associative  $k$ -algebra  $A$  (over a ground field  $k$ ) is a *Hopf RTA* (or *HRTA* in short), if:

- (1) The multiplication map  $: B_- \otimes_k H \otimes_k B_+ \rightarrow A$  is an isomorphism, where  $H, B_\pm$  are associative unital  $k$ -subalgebras of  $A$ , and  $H$  is, in addition, a commutative Hopf algebra.
- (2) The set  $G := \text{Hom}_{k\text{-alg}}(H, k)$  contains a free abelian group with finite basis  $\Delta$ , so that  $B_\pm = \bigoplus_{\lambda \in \pm \mathbb{Z}_{\geq 0} \Delta} (B_\pm)_\lambda$ . Each summand here is a finite-dimensional weight space for the (usual) adjoint action of  $H$ , and  $(B_\pm)_0 = k$ .
- (3) There exists an anti-involution  $i$  of  $A$ , so that  $i|_H = \text{id}|_H$ .

This is a large family of algebras, that are widely studied in representation theory. Examples (when  $\text{char}(k) = 0$ ) are  $\mathfrak{U}\mathfrak{g}$  for  $\mathfrak{g}$  a semisimple, symmetrizable Kac-Moody, (centerless Virasoro), or (centerless) extended Heisenberg Lie algebra. Other examples include quantum groups  $U_q(\mathfrak{g})$  or (quantized) infinitesimal Hecke algebras over  $\mathfrak{sl}_2$ .

In all these examples, we claim:

**Lemma 2.4.**  $\Gamma_A \subset \Gamma_H$ .

The same happens if we work with *skew group rings* over  $A$ , for example, wreath products  $S_n \wr A := A^{\otimes n} \rtimes S_n$ ; in this case we replace  $H$  in the above lemma, by the Hopf subalgebra  $H^{\otimes n} \rtimes S_n$  of  $S_n \wr A$ .

*Proof.* Use Proposition 2.2, with  $V = N_- + N_+$ , where  $N_{\pm}$  are the augmentation ideals in  $B_{\pm}$ . Hence  $\Gamma_A \subset \Gamma_H$  (since each weight must also kill  $[A, A] \cap H$ ).  $\square$

**Example 9** (Finite-dimensional pointed Hopf algebras). Assume that

- (1)  $R = \overline{k(R)}$  is an algebraically closed field of characteristic zero.
- (2)  $H$  is a finite-dimensional pointed Hopf algebra over  $R$ .
- (3)  $G(H)$  is a (finite) abelian group of order coprime to 210.

Then by the Classification Theorem 0.1 of [3],  $H$  is generated by  $G(H)$  and some skew-primitive generators  $\{x_i\}$ , that satisfy  $gx_i g^{-1} = \chi_i(g)x_i$  for all  $i$  and all  $g \in G(H)$ . Since  $\chi_i \neq \varepsilon \forall i$  by [3, Equation (0.1)], hence Proposition 2.2 again reduces the computations here, to the grouplike case:  $\Gamma_H \subset \Gamma_{G(H)}$ .

### 3. SKEW-PRIMITIVE ELEMENTS

We now mention various generalizations of primitive elements.

**Definition 3.1.** An element  $h \in H$  is *skew-primitive* if  $\Delta(h) = g \otimes h + h \otimes g'$  for grouplike  $g, g' \in G(H)$ . Denote the set of such elements by  $H_{g, g'}$ . Then  $g - g' \in H_{g, g'} \cap H_{g', g}$ , and  $H_{1, 1} = H_{\text{prim}}$ .

We say that  $h$  is *pseudo-primitive* (resp. *almost primitive*) with respect to  $\Pi$  if, moreover,  $g^{-1}g' \in G_{\Pi}(H)$  (resp.  $g, g' \in G_{\Pi}(H)$ ). In future, we do not specify  $\Pi$  because it is part of the given data.

Thus,  $\{\text{skew-prim.}\} \supset \{\text{pseudo-prim.}\} \supset \{\text{almost prim.}\} \supset \{\text{prim.}\}$ .

**Lemma 3.1.** *Say  $h \in H_{g, g'}$  as above.*

- (1) *The set  $\{\gamma \in \Gamma : \gamma(h) = 0\}$  is a subgroup of  $\Gamma$ .*
- (2)  *$\varepsilon(h) = 0$  and  $S(h) = -g^{-1}h(g')^{-1} \in H_{(g')^{-1}, g^{-1}}$ .*
- (3) *If  $g_0$  is any grouplike element, then  $g_0h$  and  $hg_0$  are also skew-primitive. We also have  $g_0 - g_0^{-1} \in H_{g_0, g_0^{-1}} \cap H_{g_0^{-1}, g_0}$ .*
- (4) *For all  $n \geq 0$ , we also have*

$$\Delta^{(n)}(h) = \sum_{i=0}^n g^{\otimes i} \otimes h \otimes (g')^{\otimes (n-i)} \quad (3.1)$$

- (5) *For any  $\gamma \in \Pi$ , we have  $\gamma(g) \neq \gamma(g')$ , or  $\gamma(h) = 0$ , or  $\text{char}(R) | n_{\gamma}$ .*
- (6)  *$h$  is pseudo-primitive if and only if  $g^{-1}h, hg^{-1}, (g')^{-1}h, h(g')^{-1}$  are almost primitive. If  $g_0 \in G(H)$  and  $h$  is pseudo-primitive, then so are  $g_0h$  and  $hg_0$ .*

The proofs are straightforward.

Our main result here is to compute  $\Sigma_{\Pi}(g_1hg_2)$  for any  $g_1, g_2 \in G(H)$ , or equivalently by the lemma above,  $\Sigma_{\Pi}(h)$  for all skew-primitive  $h$ .

Equation (3.1) implies that if  $\gamma \in \Gamma$ , and  $\gamma(g) \neq \gamma(g')$ , then

$$\gamma^n(h) = \gamma^{\otimes n}(\Delta^{(n-1)}(h)) = \gamma(h) \cdot \frac{\gamma(g)^n - \gamma(g')^n}{\gamma(g) - \gamma(g')} \quad (3.2)$$

**3.1. The main result.** In all that follows below, we assume that  $h$  is skew-primitive, with  $\Delta(h) = g \otimes h + h \otimes g'$ .

**Theorem 3.1.** *A skew primitive  $h \in H_{g,g'}$  satisfies at least one of the following three conditions:*

- (1) *If there is  $\lambda \in \Pi$  so that  $\lambda(g), \lambda(g') \neq 1$ , then  $\Sigma_\Pi(h) = 0$ . If no such  $\lambda$  exists, then one of  $g, g'$  is in  $G_\Pi(H)$ .*
- (2) *Suppose only one of  $g, g'$  is in  $G_\Pi(H)$ , so that there exists  $\lambda \in \Pi$  with exactly one of  $\lambda(g), \lambda(g')$  equal to 1. Then  $\Sigma_\Pi(h) = \frac{|\Pi|\lambda(h)}{1 - \lambda(gg')}$ .*
- (3) *If  $\lambda(gg') = 1$  for all  $\lambda \in \Pi$ , then  $\Sigma_\Pi(h) = \sum_{\gamma \neq \varepsilon = \gamma^2} \gamma(h)$ , and  $2\Sigma_\Pi(h) = 0$ .*

**Remark 3.1.**

- (1) Thus, the expression  $\lambda(h)/(1 - \lambda(gg')) = \Sigma_\Pi(h) \in k(R)$  is independent of  $\lambda$  (as long as  $\lambda(gg') \neq 1$ ), for such  $h$ . As the proof indicates, we should really think of  $1 - \lambda(gg')$  as  $1 - \lambda(g)$  or  $1 - \lambda(g')$  (depending on which of  $g'$  and  $g$  is in  $G_\Pi(H)$ ).

Moreover, equation (3.2) implies, whenever  $\gamma^n(g) \neq \gamma^n(g')$ , that

$$\frac{\gamma^n(h)}{\gamma^n(g) - \gamma^n(g')} = \frac{\gamma(h)}{\gamma(g) - \gamma(g')}$$

Actually, both of these are manifestations of the following (easy) fact (recall that  $H^*$  is an algebra under convolution):

**Lemma 3.2.** *Given  $h \in H_{g,g'}$ , define  $N_h := \{\gamma \in H^* : \gamma(g) \neq \gamma(g')\}$ . Suppose  $\mu, \lambda \in N_h$ . Then  $(\mu * \lambda)(h) = (\lambda * \mu)(h)$  if and only if  $f_h(\mu) = f_h(\lambda)$ , where  $f_h : N_h \rightarrow k(R)$  is given by*

$$f_h(\gamma) := \frac{\gamma(h)}{\gamma(g) - \gamma(g')}$$

(In other words, weights commute at  $h$  precisely when they lie on the same “level surface” for  $f_h$ .)

- (2) Also note that if the first two parts fail to hold, then both  $g, g' \in G_\Pi(H)$ , and the final part holds. Thus, the above theorem computes  $\Sigma_\Pi(h)$  for all skew-primitive  $h$ , if  $\text{char}(R) \neq 2$  or not both of  $g, g'$  are in  $G_\Pi(H)$ . We address the case when  $\text{char}(R) = 2$  and  $g, g' \in G_\Pi(H)$ , in the next subsection.

*Proof of Theorem 3.1.*

- (1) Apply equation (2.1) to  $h$ , to get  $\Sigma_\Pi(h) = \lambda(g)\Sigma_\Pi(h) + \lambda(h)\Sigma_\Pi(g')$ . Since  $g' \notin G_\Pi(H)$ , hence the second term vanishes, and we are left with  $(1 - \lambda(g))\Sigma_\Pi(h) = 0$ . But  $\lambda(g) \neq 1$ .

Next, if no such  $\lambda \in \Pi$  exists, then  $\Pi = \Pi_g \cup \Pi_{g'}$ , where  $\Pi_g, \Pi_{g'}$  were defined before Proposition 2.1. We claim that one of the two sets is contained in the other, whence one of  $g, g'$  is in  $G_\Pi(H)$ . For

if not, then choose  $\gamma, \gamma' \in \Pi$  so that neither  $\gamma(g)$  nor  $\gamma'(g')$  equals 1 (whence  $\gamma(g') = 1 = \gamma'(g)$ ). Then one verifies that  $\gamma\gamma' \notin \Pi_g \cup \Pi_{g'}$  and this is a contradiction. Thus one of  $\Pi_g \setminus \Pi_{g'}$ ,  $\Pi_{g'} \setminus \Pi_g$  is empty.

- (2) Suppose  $g' \in G_\Pi(H)$ ,  $\lambda(g) \neq 1$  for some  $\lambda$  (the other case is similar). Now apply equation (2.1) and Proposition 2.1, and compute:

$$\Sigma_\Pi(h) = \lambda(h) \frac{\Sigma_\Pi(g) - \Sigma_\Pi(g')}{\lambda(g) - \lambda(g')} = \lambda(h) \frac{0 - |\Pi|}{\lambda(g) \cdot 1 - 1} = \frac{\lambda(h)|\Pi|}{1 - \lambda(g)\lambda(g')}$$

- (3) If  $\lambda(gg') = 1$  for any  $\lambda \in \Pi$ , then:  $\lambda^{-1}(h) = -\lambda(h)$ . Thus  $\lambda + \lambda^{-1}$  kills  $h$ , and the first equation now follows because  $\varepsilon(h) = 0$ . The second is also easy:  $2\Sigma_\Pi(h) = \sum_{\gamma \in \Pi} (\gamma(h) + \gamma^{-1}(h)) = 0$ .

□

**3.2. The characteristic 2 case.** The only case that Theorem 3.1 does not address, is when  $\text{char}(R) = 2$  and  $g, g' \in G_\Pi(H)$ . We now address this case.

**Proposition 3.1.** *Suppose  $\Pi$  is as above,  $\text{char}(R) = 2$ , and  $h \in H_{g,g'}$  is almost primitive with respect to  $\Pi$ .*

- (1) *If  $\Pi$  has odd order, then  $\Sigma_\Pi(h) = 0$ .*
- (2) *If 4 divides  $|\Pi|$ , then  $\Sigma_\Pi(h) = 0$ .*
- (3) *If  $\Pi$  has even order but  $4 \nmid |\Pi|$ , then  $\Sigma_\Pi(h) = \gamma(h)$  for any  $\gamma \in \Pi$  of order exactly 2. This may assume any nonzero value in  $R$ .*

We omit the proof, since this result is a special case of a more general theorem in general (positive) characteristic, that we state and prove later.

#### 4. SUBGROUPS AND SUBQUOTIENTS OF GROUPS OF WEIGHTS

**4.1. Subgroups associated to arbitrary elements.** In this article, we focus on computing  $\Sigma_\Pi$  at any element  $h$  in a Hopf algebra  $H$ . However, we can also carry out the following constructions:

**Definition 4.1.** We work over  $R$ .

- (1) Suppose  $H$  is an  $R$ -algebra, so that  $\Gamma = \Gamma_H$  has a group structure  $*$  on it. Define  $\Gamma_h$  to be the *stabilizer subgroup* of  $h$ , given by

$$\Gamma_h := \{\gamma \in \Gamma : (\beta * \gamma * \delta)(h) = (\beta * \delta)(h) \ \forall \beta, \delta \in \Gamma\}$$

- (2) Given a coalgebra  $H$ , and  $h \in H$ , define  $C_h$  to be the  $R$ -subcoalgebra generated by  $h$  in  $H$ .
- (3) Given a Hopf algebra  $H$ , define  $\Gamma'_h$  to be the *fixed weight monoid* of  $h$ , given by  $\Gamma'_h := \{\gamma \in \Gamma : \gamma|_{C_h} = \varepsilon|_{C_h}\}$ .

In particular,  $\gamma(h) = \varepsilon(h)$  if  $\gamma \in \Gamma_h$ .

We talk of how this allows us to consider subquotients of  $\Gamma$ , in a later subsection; but first, we make some observations involving these subgroups.

**Proposition 4.1.**

- (1) For all  $h$ ,  $\Gamma_h$  is a normal subgroup of  $\Gamma$ , and  $\Gamma'_h \subset \Gamma_h$  is a monoid closed under  $\Gamma$ -conjugation.
- (2) Given  $\{h_i : i \in I\} \subset H$ , and  $h \in \langle h_i \rangle$  (i.e., in the subalgebra generated by the  $h_i$ 's),  $\Gamma_h \supset \bigcap_{i \in I} \Gamma_{h_i}$ , and similarly for the  $\Gamma'$ s.
- (3) Given any  $h_i \in H$  (finitely many), suppose  $\Pi = \times_i \Pi_i$ , with  $\Pi_i \subset \Gamma'_{h_j}$  whenever  $i \neq j$ . Then  $\Sigma_\Pi(\mathbf{h}) = \prod_i \Sigma_{\Pi_i}(h_i)$ .

*Proof.* The first and third parts are straightforward computations. For the second part, for all  $\beta, \delta \in \Gamma$ ,  $\gamma \in \bigcap_i \Gamma_{h_i}$ , and polynomials  $p$  in the  $h_i$ 's,

$$(\beta * \gamma * \delta)(p(h_i)) = p((\beta * \gamma * \delta)(h_i)) = p((\beta * \delta)(h_i)) = (\beta * \delta)(p(h_i))$$

The outer equalities hold because weights are algebra maps.

The proof for the  $\Gamma'$ s is as follows: if  $h = p(h_i)$  as above, then since  $\Delta$  is multiplicative, hence any  $h' \in C_h$  is expressible as a polynomial in elements  $h'_j \in \cup_i C_{h_i}$  - say  $h' = q(h'_j)$ . We conclude that if  $\gamma \in \Gamma'_{h_i}$  for all  $i$ , then

$$\gamma(q(h'_j)) = q(\gamma(h'_j)) = q(\varepsilon(h'_j)) = \varepsilon(q(h'_j))$$

where once again, the outer equalities hold because weights are algebra maps. In other words,  $\gamma(h') = \varepsilon(h')$ .  $\square$

We also mention two examples; the proofs are straightforward.

**Lemma 4.1.** *If  $g \in G(H)$ , then this definition of  $\Gamma_g$  coincides with the previous one:  $\Gamma_g = \Gamma'_g = \{\gamma \in \Gamma : \gamma(g) = 1\}$ . If  $h \in H_{g,g'}$ , then  $\Gamma_h \subset \Gamma_g \cap \Gamma_{g'}$ , or  $\Gamma_h = \Gamma$ . In both cases,  $\Gamma'_h = \Gamma_h \cap \Gamma'_g \cap \Gamma'_{g'}$ .*

**4.2. Subquotients.** We now compute  $\Sigma_\Pi(h)$  for more general  $\Pi$ : given  $h \in H$ , choose any subgroup  $\Gamma'$  of  $\Gamma_h$  that is normal in  $\Gamma$ , and consider any finite subgroup  $\Pi$  of  $\Gamma/\Gamma'$ . We now choose any lift  $\tilde{\Pi}$  of  $\Pi$  to  $\Gamma$ , and define

$$\Sigma_\Pi(h) := \sum_{\gamma'' \in \tilde{\Pi}} \gamma''(h), \quad \Pi^\circ := \{\gamma \in \Gamma : (\gamma + \Gamma') \in \Pi \subset \Gamma/\Gamma'\}$$

The following result is used later;  $\tilde{\Pi}$  occurs in its proof (and also later).

**Lemma 4.2.** *Setup (i.e.,  $h, \Gamma_h, \Gamma', \Pi, \Pi^\circ$ ) as above.*

- (1)  $\Sigma_\Pi(h)$  is well-defined.
- (2) If a subgroup  $\Gamma'' \subset \Gamma_h$  is normal in  $\Pi^\circ$  (e.g.,  $\Gamma'' = \Gamma'$ ,  $\Gamma_h \cap \Pi^\circ$ ), then

$$\Sigma_{\Pi^\circ}(h) = |\Gamma''| \Sigma_{\Pi^\circ/\Gamma''}(h)$$

While a special case of the second part is that  $\Sigma_{\Pi^\circ}(h) = |\Gamma'| \Sigma_\Pi(h)$ , we really use the result when  $\Gamma'$  is itself finite, and we replace  $\Pi^\circ$  by  $\Pi$ . The equation is then used to compute  $\Sigma_\Pi(h)$ .

**4.3. Pseudo-primitive elements.** For the rest of this paper,  $H$  is an  $R$ -Hopf algebra, unless stated otherwise. If  $h$  is grouplike or (pseudo)primitive, then it is easy to see that  $\Sigma_{\Pi}(h) = |[\Pi, \Pi]| \Sigma_{\Pi_{ab}}(h)$ , where  $\Pi_{ab} := \Pi/[\Pi, \Pi]$  (we show the pseudo-primitive case presently). Thus, in the grouplike case, the question of whether or not  $h \in G_{\Pi}(H)$  reduces to evaluating (any lift of a set of) generators of (the finite abelian group)  $\Pi_{ab}$ , at  $h$ .

**Proposition 4.2.** *Suppose  $h \in H_{g,g'}$  is pseudo-primitive with respect to  $\Pi$ .*

- (1) *Then  $(\gamma * \nu)(h) = (\nu * \gamma)(h) = \nu(g)\gamma(h) + \gamma(g)\nu(h)$  for all  $\gamma, \nu \in \Pi$ .*
- (2)  *$\Gamma'_h \supset [\Pi, \Pi]$ .*
- (3) *For any  $m \geq 0$ , we have  $\gamma^{*m}(h) = m\gamma(g)^{m-1}\gamma(h)$  if  $\gamma \in \Pi$ . In particular, if  $\text{char}(R) = p$  is prime, then  $\gamma^{*p}(h) = 0 \forall \gamma \in \Pi$ .*

*Proof.* The first and last parts are by definition and induction respectively. As for the second part, one shows the following computation for any skew-primitive  $h \in H_{g,g'}$ , and  $\beta, \beta' \in \Gamma_H$ :

$$\begin{aligned} (\beta * \beta' * \beta^{-1} * (\beta')^{-1})(h) &= \beta(h(g')^{-1})(1 - \beta'(g(g')^{-1})) \\ &\quad + \beta'(h(g')^{-1})(\beta(g(g')^{-1}) - 1) \end{aligned}$$

using Lemma 3.1. Since  $h$  is pseudo-primitive with respect to  $\Pi$ , this shows that every generator (and hence element)  $\lambda$  of  $[\Pi, \Pi]$  satisfies:  $\lambda(g) = \lambda(g') = 1$  and  $\lambda(h) = 0$ . Since  $C_h = Rh + Rg + Rg'$ , these imply that  $\lambda \in \Gamma'_h$ .  $\square$

An easy consequence of Propositions 4.1 and 4.2 is

**Corollary 4.1.** *If  $h \in H$  is (in the subalgebra) generated by grouplike and pseudo-primitive elements (with respect to  $\Pi$ ), then  $\Gamma'_h \supset [\Pi, \Pi]$ .*

Also note that given some  $h \in H$ , we can compute  $\Sigma_{\Pi}(h)$  for more general  $\Pi$ , and hence the results in this paper can be generalized; however, we stick to our original setup when  $\Pi \subset \Gamma$  (i.e.,  $\Gamma' = \{\varepsilon\}$ ). We need to note this (general) case, though, because we use it below.

We conclude by specifying more precisely, what we mean by  $\Sigma_{\Pi}(h_1 \dots h_n)$  for “pseudo-primitive”  $h_i$ ’s, when  $\Pi$  is a subquotient of  $\Gamma$  as above. Thus, we start with some skew-primitive  $h_i$ ’s, then let  $\Pi$  be a finite subgroup of  $\Gamma/\Gamma'$ , for some subgroup  $\Gamma' \subset \cap_i \Gamma_{h_i}$  that is normal in  $\Gamma$ . Moreover, if  $\Delta(h_i) = g_i \otimes h_i + h_i \otimes g'_i$ , then we further assume that  $\gamma(g_i) = \gamma(g'_i)$ , for all elements  $\gamma$  of the subgroup  $\Pi^{\circ}$  (defined above).

This is what we mean in the case of general  $\Pi$ , when we say that  $g_i, g'_i \in G_{\Pi}(H)$  - i.e., that the  $h_i$ ’s are pseudo-primitive (with respect to  $\Pi$ ). Similarly, in saying that the  $h_i$ ’s are almost primitive with respect to  $\Pi$ , we mean that  $\gamma(g_i) = \gamma(g'_i) = 1 \forall \gamma \in \Pi^{\circ}$ .

## 5. PRODUCTS OF SKEW-PRIMITIVE ELEMENTS

We now mention some results on (finite) products of skew-primitive elements and grouplike elements. From now on,  $\Pi$  denotes a finite subgroup

of  $\Gamma$  and not a general subquotient; however, in the next section, we need to use a subquotient  $\Phi$  of this  $\Pi$ .

Since the set of skew-primitive elements is closed under multiplication by grouplike elements, any “monomial” in them can be expressed in the form  $\mathbf{h} = \prod_i h_i$ . The related “grouplike” elements that would figure in the computations, are  $\mathbf{g} = \prod_i g_i$  and  $\mathbf{g}' = \prod_i g'_i$ .

**Standing Assumption 5.1.** For this section and the next two, we assume that  $h_i \in H_{g_i, g'_i}$  for all (finitely many)  $i$ .

We first mention some results that hold in general.

**Proposition 5.1.** *If  $\lambda(h_i) = 0 \forall i$  for some  $\lambda \in \Pi$ , and (at least) one of  $\lambda(\mathbf{g}), \lambda(\mathbf{g}')$  is not 1, then  $\Sigma_\Pi(\mathbf{h}) = 0$ .*

*Proof.* If  $\lambda(h_i) = 0$  then  $\lambda^m(h_i) = 0$  for all  $i, m$ . We now choose a set  $\mathcal{B}$  of coset representatives for  $\langle \lambda \rangle$  in  $\Pi$ , and assume that  $\lambda(\mathbf{g}') \neq 1$  (the other case is similar). Then we compute:

$$\Sigma_\Pi(\mathbf{h}) = \sum_{\beta \in \mathcal{B}, \gamma \in \langle \lambda \rangle} \prod_i (\beta(g_i)\gamma(h_i) + \beta(h_i)\gamma(g'_i))$$

Since  $\gamma(h_i) = 0$  for all  $\gamma \in \langle \lambda \rangle$  and all  $i$ , hence the entire product in the summand collapses, to give  $\sum_{\beta \in \mathcal{B}} \beta(\mathbf{h}) \cdot \Sigma_{\langle \lambda \rangle}(\mathbf{g}')$ . But now the second factor vanishes by our assumption (and Proposition 2.1).  $\square$

Next, if we know  $\Sigma_\Pi(\prod_{i=1}^n h_i)$ , for all skew-primitive  $h_i$ 's, then we can evaluate the product of  $(n+1)$  such  $h$ 's in some cases. The following result relates  $\Sigma_\Pi$ -values of strings to the  $\Sigma_\Pi$ -values of proper substrings (with skew-primitive “letters”), that are “corrected” by grouplike elements. The proof is that both equations below follow by evaluating equation (2.1) at  $\mathbf{h}$ .

**Proposition 5.2.** *Suppose one of  $\mathbf{g}, \mathbf{g}'$  is not in  $G_\Pi(H)$ . Thus, if for some  $\lambda \in \Pi$ , we have  $\lambda(\mathbf{g}) \neq 1$ , or respectively  $\lambda(\mathbf{g}') \neq 1$ , then*

$$\Sigma_\Pi(\mathbf{h}) = \frac{\sum_{\nu \in \Pi} \prod_i (\lambda(g_i)\nu(h_i) + \lambda(h_i)\nu(g'_i)) - \lambda(\mathbf{g})\Sigma_\Pi(\mathbf{h})}{1 - \lambda(\mathbf{g})}$$

or respectively,

$$\Sigma_\Pi(\mathbf{h}) = \frac{\sum_{\nu \in \Pi} \prod_i (\nu(g_i)\lambda(h_i) + \nu(h_i)\lambda(g'_i)) - \Sigma_\Pi(\mathbf{h})\lambda(\mathbf{g}')}{1 - \lambda(\mathbf{g}')}$$

Note, here, that both numerators on the right side have an  $\Sigma_\Pi(\mathbf{h})$  in them, which cancels the only such term present in the summations. Thus, what we are left with in either case, are linear combinations of  $\Sigma_\Pi$ -values of “corrected” proper substrings, with coefficients of the form  $\lambda(\prod g_j h_k)$ .

Also note that if we knew the  $\Sigma_\Pi$ -values of all “corrected” proper substrings, and  $\text{char}(R) \neq 2$ , then the two propositions, one above and one below, can be used, for instance, to compute  $\Sigma_\Pi$  at all monomials of odd length (in skew-primitive elements).

The statement and proof of the following result are essentially the same as those of the last part of Theorem 3.1 above.

**Proposition 5.3.** *Suppose  $\mathbf{g}\mathbf{g}' \in G_\Pi(H)$ , and  $\text{char}(R) = 2$  if the number of  $h_i$ 's is even. Then  $\Sigma_\Pi(\mathbf{h}) = \sum_{\gamma \neq \varepsilon = \gamma^2} \gamma(\mathbf{h})$ , and  $2\Sigma_\Pi(\mathbf{h}) = 0$ .*

This is because we once again get that  $\lambda^{-1}(\mathbf{h}) = -\lambda(\mathbf{h}) \forall \lambda$ .

The next result in this subsection is true for almost all values of  $\text{char } R$ . The proof is immediate from the penultimate part of Lemma 3.1 above.

**Proposition 5.4.** *If  $\text{char}(R) \nmid |\Pi|$  and  $g_i^{-1}g'_i \in G_\Pi(H)$  (i.e.,  $h_i$  is pseudo-primitive) for some  $i$ , then  $\gamma(h_i) = 0 \forall \gamma \in \Pi$ . In particular,  $\Sigma_\Pi(\mathbf{h}) = 0$ .*

We conclude this section with one last result - in characteristic  $p$ .

**Theorem 5.1.** *Suppose  $\text{char}(R) = p > 0$ , and  $h_i \in H_{g_i, g'_i}$  for all  $i$ . Choose and fix a  $p$ -Sylow subgroup  $\Pi_p$  of  $\Pi$ .*

- (1) *Then each  $h_i$  is almost primitive with respect to  $\Pi_p$ .*
- (2) *If  $\Pi_p$  contains an element of order  $p^2$ , then  $\Sigma_\Pi(\mathbf{h}) = \Sigma_{\Pi_p}(\mathbf{h}) = 0$ .*

We also see later, that  $\Sigma_{\Pi_p}(\mathbf{h}) = 0$  if  $\Pi_p \not\cong (\mathbb{Z}/p\mathbb{Z})^k$  for any  $k > 0$ .

*Proof.*

- (1) If  $z^p = 1$  in  $R$ , then  $(1 - z)^p = 1 - z^p \pmod{p} = 0$  in  $R$ . Since  $R$  is an integral domain,  $z = 1$ . Now assume that  $|\Pi_p| = p^f$ . Thus for each  $\gamma \in \Pi_p$ , we have  $\gamma^{*p^f} = \varepsilon$ , whence  $\gamma(g_i)^{p^f} = \gamma(g'_i)^{p^f} = 1$  for all  $i$ . Successively set  $z = \gamma(g_i)^{p^t}$ , for  $t = f - 1, f - 2, \dots, 1, 0$ . Hence  $\gamma(g_i) = 1$  for all  $i$ ; the other case is the same.
- (2) Next, if  $\lambda$  has order  $p^2$ , then choose a set  $\mathcal{B}$  of coset representatives for  $\langle \lambda \rangle$  in  $\Pi$ , and compute using Proposition 4.2 above (since all  $h_i$ 's are pseudo/almost primitive with respect to  $\Pi_p$ , hence for  $\langle \lambda \rangle$ ):

$$\begin{aligned} \Sigma_\Pi(\mathbf{h}) &= \sum_{\beta \in \mathcal{B}} \sum_{j=0}^{p^2-1} \prod_{i=1}^n (\beta * \lambda^{*j})(h_i) \\ &= \sum_{\beta \in \mathcal{B}} \sum_{j=0}^{p^2-1} \prod_{i=1}^n (\beta(h_i)\lambda^j(g_i) + j\beta(g_i)\lambda(g_i)^{j-1}\lambda(h_i)) \end{aligned}$$

Call the factor in the product  $a_{ij}$  (it really is  $a_{\beta, i, j}$ ). Then we observe that  $a_{i, p+j} = \lambda^p(g_i)a_{ij}$  for all  $i, j$ , whence  $a_{i, kp+j} = \lambda^{kp}(g_i)a_{ij}$ . Therefore for any  $\beta \in \mathcal{B}$ , we can take  $\sum_{k=0}^{p-1} \lambda(\mathbf{g})^{kp}$  out of the summand. But  $\lambda(\mathbf{g})^p = 1$ , so every  $\beta$ -summand vanishes. □

## 6. SPECIAL CASE - ABELIAN GROUP OF WEIGHTS

In this section, we focus on the special case where the group  $\Pi$  of weights (that we sum over) is abelian. We then evaluate  $\Sigma_{\Pi}(\mathbf{h})$  as above.

**Definition 6.1.**

- (1) For all  $n \in \mathbb{N}$ , define  $[n] := \{1, 2, \dots, n\}$ .
- (2) Given  $I \subset [n]$ , define  $g_I := \prod_{i \in I} g_i$ , and similarly define  $g'_I, h_I$ .
- (3) Define  $\Pi_p$  to be any fixed  $p$ -Sylow subgroup of  $\Pi$  if  $\text{char}(R) = p > 0$ , and  $\{\varepsilon\}$  otherwise. Also choose and fix a ‘‘complementary’’ subgroup  $\Pi'$  to  $\Pi_p$  in  $\Pi$  (if  $\Pi$  is abelian), i.e.,  $|\Pi_p| \cdot |\Pi'| = |\Pi|$ . (And if  $\text{char}(R) = 0$ , set  $\Pi' := \Pi$ .)

We present two results here. The first is (nontrivial only) when  $\text{char}(R)$  divides the order of  $\Pi$ , and the second (which really is the main result) is when it does not.

**Theorem 6.1.** *Suppose  $\Pi$  is abelian; let  $\Pi_p, \Pi'$  be as above. Let  $J \subset [n]$  be the set of  $i$ 's such that  $h_i$  is pseudo-primitive with respect to  $\Pi$ . Then*

$$\Sigma_{\Pi}(\mathbf{h}) = \Sigma_{\Pi'}(g_J h_{[n] \setminus J}) \cdot \Sigma_{\Pi_p}(h_J)$$

(Thus, if  $\text{char}(R) = 0$ , or  $0 < \text{char}(R) \nmid |\Pi|$ , then  $\Sigma_{\Pi_p}(h_J) = \varepsilon(h_J) = 0$ , whence  $\Sigma_{\Pi}(\mathbf{h}) = 0$  too.)

*Proof.* First note that  $\lambda(g_i) = \lambda(g'_i) = 1$  for all  $\lambda \in \Pi_p$  and all  $i$ , by Theorem 5.1 above. Since  $\Pi$  is abelian, every  $\gamma \in \Pi$  is uniquely expressible as  $\gamma = \beta * \lambda$  with  $\beta \in \Pi', \lambda \in \Pi_p$ . We compute  $\gamma(h_i)$  in both cases:  $i \in J$  and  $i \notin J$ .

Let us first consider the case when  $i \in J$ . Then  $\beta(h_i) = 0$  for all  $\beta \in \Pi'$ , by Proposition 5.4. Thus,  $\gamma(h_i) = \beta(g_i)\lambda(h_i) + \beta(h_i)\lambda(g_i) = \beta(g_i)\lambda(h_i)$ .

Now suppose  $i \notin J$ . Choose  $\gamma \in \Pi$  so that  $\gamma(g_i) \neq \gamma(g'_i)$ . Then  $(\gamma * \lambda)(h_i) = (\lambda * \gamma)(h_i)$ , which leads (upon simplifying) to

$$\lambda(h_i)(\gamma(g_i) - \gamma(g'_i)) = \gamma(h_i)(\lambda(g_i) - \lambda(g'_i))$$

But  $\lambda(g_i) = \lambda(g'_i)$ , and  $\gamma(g_i) \neq \gamma(g'_i)$ , so  $\lambda(h_i) = 0$  for all  $\lambda \in \Pi_p$ . Hence:

$$\gamma(h_i) = (\beta * \lambda)(h_i) = \beta(g_i)\lambda(h_i) + \beta(h_i)\lambda(g_i) = \beta(h_i)$$

We are now ready to complete the proof.

$$\begin{aligned} \Sigma_{\Pi}(\mathbf{h}) &= \sum_{\beta \in \Pi'} \prod_{\lambda \in \Pi_p} \prod_{i \in J} \beta(g_i)\lambda(h_i) \cdot \prod_{i \notin J} \beta(h_i) \\ &= \sum_{\beta \in \Pi', \lambda \in \Pi_p} \beta(g_J h_{[n] \setminus J})\lambda(h_J) = \Sigma_{\Pi'}(g_J h_{[n] \setminus J}) \cdot \Sigma_{\Pi_p}(h_J) \end{aligned}$$

as claimed.  $\square$

We compute  $\Sigma_{\Pi_p}(\mathbf{h})$  in a later section. For now, we mention how to compute the other factor. For this, we need Lemma 3.2 above.

**Theorem 6.2.** *Suppose  $\text{char}(R) \nmid |\Pi|$ , and no  $h_i$  is pseudo-primitive with respect to (the abelian group of weights)  $\Pi$ . For each  $i$ , let  $f_i \in k(R)$  denote  $\beta(h_i)/(\beta(g_i) - \beta(g'_i))$  for some  $\beta \in N_{h_i} \cap \Pi$ . Also define*

$$S = \{I \subset [n] : g_I g'_{[n] \setminus I} \in G_\Pi(H)\}$$

Then  $\Sigma_\Pi(\mathbf{h}) = (-1)^n |\Pi| \prod_{i=1}^n f_i \cdot \sum_{I \in S} (-1)^{|I|}$ .

Note that if  $n = 1$ , this is a special case of the first two parts of Theorem 3.1 above. Moreover, if some  $h_i$  is pseudo-primitive with respect to  $\Pi$ , then  $\Sigma_\Pi(\mathbf{h}) = 0$  from Proposition 5.4 above.

*Proof.* Let us fix some generators  $\beta_1, \dots, \beta_k$  of  $\Pi$  (by the structure theory of finite abelian groups), so that  $\Pi = \bigoplus_{j=1}^k \mathbb{Z}\beta_j$ . Then (by assumption,) for each  $i$  there is at least one  $j$  so that  $\beta_j(g_i) \neq \beta_j(g'_i)^{-1}$ .

Let us now fix  $i$ , and compute  $\beta(h_i)$  for arbitrary  $\beta \in \Pi$ . Suppose  $N'_i$  indexes the set of  $\beta_j$ 's that are in  $N_{h_i}$ ; then we can write  $\beta = \beta' + \sum_{j \in N'_i} r_j \beta_j$  for some  $r_j \geq 0$  and  $\beta' \in \bigoplus_{j \notin N'_i} \mathbb{Z}\beta_j$ .

We first note by Proposition 5.4 for  $\Pi \leftrightarrow \bigoplus_{j \notin N'_i} \mathbb{Z}\beta_j$  that  $\beta'(h_i) = 0$ . So if  $\beta'' = (\beta')^{-1}\beta \in \Pi$ , then

$$\beta(h_i) = \beta'(g_i)\beta''(h_i) + \beta'(h_i)\beta''(g_i) = \beta'(g_i)\beta''(h_i)$$

It remains to compute the last factor above. We show the following claim below, first using it to finish the proof.

**Claim.** Say  $\beta'' = \sum_{j \in N'_i} r_j \beta_j$ . Then  $\beta''(h_i) = (\beta''(g_i) - \beta''(g'_i))f_i$ , where  $f_i$  is defined in the statement of the theorem.

*Proof of the theorem, modulo the claim.* By the claim, we therefore get that

$$\beta(h_i) = \beta'(g_i)(\beta''(g_i) - \beta''(g'_i))f_i = (\beta(g_i) - \beta(g'_i))f_i$$

(for all  $i$ ) since  $\beta'(g_i) = \beta'(g'_i)$  by pseudo-primitivity. Using the notation that  $\beta_{\mathbf{r}} = \sum_{j=1}^k r_j \beta_j \in \Pi$ , we now compute  $\Sigma_\Pi(\mathbf{h})$  to be

$$\begin{aligned} &= \sum_{\mathbf{r}} \prod_{i=1}^n (\beta_{\mathbf{r}}(g_i) - \beta_{\mathbf{r}}(g'_i))f_i = \prod_{i=1}^n f_i \cdot \sum_{\mathbf{r}} \sum_{I \subset [n]} (-1)^{n-|I|} \beta_{\mathbf{r}} \left( \prod_{i \in I} g_i \prod_{j \notin I} g'_j \right) \\ &= (-1)^n \prod_{i=1}^n f_i \cdot \sum_{I \subset [n]} (-1)^{|I|} \Sigma_\Pi(g_I g'_{[n] \setminus I}) = (-1)^n \prod_{i=1}^n f_i \cdot \sum_{I \subset [n]} (-1)^{|I|} \delta_{I \in S} |\Pi| \end{aligned}$$

by Proposition 2.1, where the last  $\delta$  is 1 if  $I \in S$ , and 0 otherwise.  $\square$

To complete the proof, we now show the claim.

*Proof of the claim.* By equation (3.2), and Lemma 3.2, we see that for all  $j \in N'_i$ , we have

$$\beta_j^{*r_j}(h_i) = (\beta_j(g_i)^{r_j} - \beta_j(g'_i)^{r_j})f_i$$

Suppose without loss of generality, that we relabel the set  $\{\beta_j : j \in N'_i\}$  as  $\{\beta_1, \dots, \beta_m\}$  (i.e., relabel the generators  $\beta_j$  of  $\Pi$  so that these are before the others). We now compute the expression. Using the above equation,

$$\beta''(h_i) = \left( \sum_{j=1}^m r_j \beta_j \right) (h_i) = \sum_{j=1}^m \prod_{l < j} \beta_l(g_i)^{r_l} \cdot (\beta_j(g_i)^{r_j} - \beta_j(g'_i)^{r_j}) f_i \cdot \prod_{l > j} \beta_l(g'_i)^{r_l}$$

and this telescopes to  $f_i \cdot \prod_j \beta_j(g_i)^{r_j} - f_i \cdot \prod_j \beta_j(g'_i)^{r_j} = (\beta''(g_i) - \beta''(g'_i))f_i$ , as claimed.  $\square$

## 7. PRODUCTS OF PSEUDO-PRIMITIVE ELEMENTS - POSITIVE CHARACTERISTIC

We now mention results for pseudo-primitive elements  $h_i$  (and not necessarily abelian  $\Pi$ ) in prime characteristic; note that for almost all characteristics (including zero), Proposition 5.4 above says that  $\Sigma_{\Pi}(\mathbf{h}) = 0$ . Before we move on to the positive case, we need a small result.

**Lemma 7.1.** *Given  $f \in \mathbb{N}$  and a prime  $p > 0$ , define  $\varphi_p(f) = \varphi_p(f) := \sum_{i=0}^{p-1} i^f$ . If  $f > 0$ , then  $\varphi(f) \not\equiv 0 \pmod p$  if and only if  $(p-1) \mid f$ , and in this case,  $\varphi(p-1) = p-1 \equiv -1 \pmod p$ .*

*Proof.* Let  $g$  be any cyclic generator of (the finite cyclic group)  $(\mathbb{Z}/p\mathbb{Z})^\times$ . Then  $\sum_{i=1}^{p-1} i^f \equiv \sum_{j=1}^{p-1} g^{jf} \pmod p$ . Now if  $(p-1) \mid f$ , then each summand is 1, and we get  $p-1 \pmod p$ . Otherwise,  $g^f$  is not 1, and its powers add up to 0 (by the geometric series formula).  $\square$

**7.1. Preliminaries.** Recall that a pseudo-primitive element is any  $h \in H_{g,g'}$  so that  $g^{-1}g' \in G_{\Pi}(H)$ . We now need some terminology. Note by Hall's theorems, that a finite group  $\Phi$  is solvable if and only if it contains Hall subgroups of all possible orders (e.g., see [2, §11]). So if  $|\Phi| = p^k \cdot m$  with  $p \nmid m$ , let  $\Phi_m$  be any Hall subgroup of order  $m$ .

**Definition 7.1.** ( $p > 0$  a fixed prime.) Given a finite solvable group  $\Phi$ , denote by  $\Phi_p, \Phi'$  respectively, any  $p$ -Sylow subgroup and any Hall subgroup of order  $|\Phi|/|\Phi_p|$ . (From above, we mean  $\Phi' = \Phi_m$ .)

For the rest of this section,  $\text{char}(R) = p > 0$ ; also fix  $n$ , the number of  $h_i$ 's.

**Proposition 7.1.** *Suppose, given skew-primitive  $h_i$ 's, that  $\Phi \subset \Gamma/\Gamma'$  is a finite solvable subquotient of  $\Gamma$  (as in a previous section) with respect to which every  $h_i$  is pseudo-primitive. Then  $\Sigma_{\Phi}(\mathbf{h}) = \Sigma_{\Phi'}(\mathbf{g}) \cdot \Sigma_{\Phi_p}(\mathbf{h})$ .*

*Proof.* We first claim that the “set-product”  $\Phi' \Phi_p := \{\beta * \lambda : \beta \in \Phi', \lambda \in \Phi_p\}$  equals the entire group  $\Phi$ . Next, if  $\beta \in \Phi'$ , then  $\beta(h) = 0$  for any pseudo-primitive  $h \in H_{g,g'}$ , because if  $|\Phi'| = m \neq 0 \pmod p$ , then  $\beta^{*m} = \varepsilon \in \Gamma' \subset \Gamma_h$ , whence  $0 = \varepsilon(h) = \beta^{*m}(h) = m\beta(g)^{m-1}\beta(h)$ . Therefore we compute:

$$\begin{aligned} \Sigma_{\Phi}(\mathbf{h}) &= \sum_{\beta \in \Phi'} \prod_{\lambda \in \Phi_p, i=1}^n (\beta * \lambda)(h_i) = \sum_{\beta \in \Phi'} \prod_{\lambda \in \Phi_p, i=1}^n \beta(g_i) \lambda(h_i) \\ &= \sum_{\beta \in \Phi'} \prod_{i=1}^n \beta(g_i) \cdot \sum_{\lambda \in \Phi_p} \prod_{i=1}^n \lambda(h_i) = \Sigma_{\Phi'}(\mathbf{g}) \Sigma_{\Phi_p}(\mathbf{h}) \end{aligned}$$

as claimed.  $\square$

**Remark 7.1.** The above proposition thus holds for *any* group  $\Phi$ , such that some  $p$ -Sylow subgroup  $\Phi_p$  has a complete set of coset representatives, none of whom has order divisible by  $p$ . Obvious examples are abelian groups or groups of order  $p^a q^b$  for primes  $p \neq q$  (but these are solvable by Burnside’s Theorem).

Also note that the above sum is independent of the choices of  $\Phi_p, \Phi'$ .

The next result is crucial in computing  $\Sigma_{\Pi}(\mathbf{h})$ , and uses subquotients of  $\Pi$ .

**Theorem 7.1.** *Given  $\Pi \subset \Gamma$ , suppose  $h_i \in H$  is pseudo-primitive with respect to  $\Pi$  for all  $i$ . Define  $\Phi = \Pi_{ab} := \Pi/[\Pi, \Pi]$ . Then*

$$\Sigma_{\Pi}(\mathbf{h}) = |[\Pi, \Pi]| \cdot \Sigma_{\Phi'}(\mathbf{g}) \cdot \Sigma_{\Phi_p}(\mathbf{h}) \quad (7.1)$$

For instance, if every  $h_i$  was almost primitive, then  $\Sigma_{\Phi'}(\mathbf{g}) = [\Phi : \Phi_p]$ .

*Proof.* At the outset, we note that  $\Sigma_{\Phi'}(\mathbf{g})$  and  $\Sigma_{\Phi_p}(\mathbf{h})$  make sense because of Corollary 4.1 and Proposition 4.1 above. Now, the proof is in two steps; each step uses a previously unused result above.

**Step 1.** We claim that  $\Sigma_{\Pi}(\mathbf{h}) = |[\Pi, \Pi]| \Sigma_{\Phi}(\mathbf{h})$ . This follows immediately from Lemma 4.2, where we replace  $h, \Gamma'', \Pi^\circ$  by  $\mathbf{h}, [\Pi, \Pi], \Pi$  respectively.

The only thing we need to check, is that the above replacements are indeed valid. Since  $[\Pi, \Pi]$  is normal in  $\Pi$ , we only need to check that  $[\Pi, \Pi] \subset \Gamma_{\mathbf{h}}$ . But this follows from Corollary 4.1 and Proposition 4.1 above.

**Step 2.** The proof is now complete, if we invoke Proposition 7.1 above.  $\square$

We conclude the preliminaries with one last result - for *skew*-primitive elements in general.

**Proposition 7.2.** *If  $\Pi_p \not\cong (\mathbb{Z}/p\mathbb{Z})^k$  for any  $k > 0$ , then  $\Sigma_{\Pi_p}(\mathbf{h}) = 0$ .*

*Proof.* By Theorem 5.1, the  $h_i$ ’s are almost primitive with respect to  $\Pi_p$ . Hence we can invoke equation (7.1) above, if we replace  $\Pi$  by  $\Pi_p$ . Now, if  $\Pi_p$  is not abelian, then  $|[\Pi_p, \Pi_p]| > 1$ , hence is a power of  $p$ , whence the right-hand side vanishes. Next, if  $\Pi_p$  is abelian, but contains an element of

order  $p^2$ , then  $\Sigma_{\Pi_p}(\mathbf{h}) = 0$  by Theorem 5.1 again. Therefore  $\Pi_p \cong (\mathbb{Z}/p\mathbb{Z})^k$  for some  $k$ . If  $k = 0$ , then  $\Pi_p = \{\varepsilon\}$ , and  $\varepsilon(\mathbf{h}) = 0$ .  $\square$

**7.2. The main results - pseudo-primitive elements.** The following result now computes  $\Sigma_{\Pi}(\mathbf{h})$  (for pseudo-primitive  $h_i$ 's) in most cases in prime characteristic that are “nonabelian”. For the “abelian” case, we appeal to Theorem 7.3 below - and mention at the outset, that it is only for *almost* primitive (and not merely pseudo-primitive) elements, that we get a much clearer picture - as its last part shows.

**Theorem 7.2.** *Suppose  $\text{char}(R) = p \in \mathbb{N}$ ,  $\Pi_p$  is any (fixed)  $p$ -Sylow subgroup of  $\Pi$ , and every  $h_i$  is pseudo-primitive (with respect to  $\Pi$ ).*

- (1)  $\Sigma_{\Pi}(\mathbf{h}) = 0$  if  $\Pi_p$ 
  - (a) is trivial,
  - (b) contains an element of order  $p^2$ , or
  - (c) intersects  $[\Pi, \Pi]$  nontrivially.

*This last part includes the cases when  $\Pi_p$*

- (d) is not abelian,
- (e) does not map isomorphically onto (some)  $\Phi_p$ , via (the restriction of) the quotient map  $\pi : \Pi \rightarrow \Phi = \Pi/[\Pi, \Pi]$ , or
- (f) has size strictly greater than  $\Phi_p$ .

- (2) *Otherwise  $\Pi_p \cong \Phi_p \cong (\mathbb{Z}/p\mathbb{Z})^k$  for some  $k > 0$ , and then  $\Sigma_{\Pi}(\mathbf{h}) = |[\Pi, \Pi]| \cdot \Sigma_{\Phi}(\mathbf{g}) \cdot \Sigma_{\Pi_p}(\mathbf{h})$ .*

**Remark 7.2.** Any finite abelian group of exponent  $p$  is of the form  $(\mathbb{Z}/p\mathbb{Z})^k$ , hence one part of the second statement is clear. Moreover, every subquotient of such a group is of the same form. Finally (especially when all of the  $h_i$ 's are almost primitive with respect to  $\Pi$ ), the cases that remain reduce to computing  $\Sigma_{\Pi_p}(\mathbf{h})$ , and when  $\Pi_p \cong (\mathbb{Z}/p\mathbb{Z})^k$ ; we address this below.

*Proof.* The second part follows from the first part, the remarks above, and equation (7.1). We now show the first part.

- (a) If  $\Pi_p$  is trivial, then  $p \nmid |\Pi|$ , and we are done by Proposition 5.4.
- (b) This has been done in Theorem 5.1 above.
- (c) Now suppose that  $[\Pi, \Pi] \cap \Pi_p \neq \emptyset$ . Then  $[\Pi, \Pi]$  contains an element of order  $p$ , whence  $p$  divides  $|[\Pi, \Pi]|$ . Now use equation (7.1).

It remains to show how this last includes the remaining cases.

- (d) First, if  $\Pi_p$  is nonabelian, then  $[\Pi_p, \Pi_p]$  is a nontrivial subgroup of the  $p$ -group  $\Pi_p$ . In particular,  $\Pi_p$  intersects  $[\Pi, \Pi]$ .

- (e) Next, note that  $\pi(\Pi_p)$  is a  $p$ -group in  $\Phi$ , and  $|\Pi_p| \geq |\Phi_p|$  (since  $|\Phi|$  divides  $|\Pi|$ ). Hence  $\Pi_p$  does not map isomorphically onto (some)  $\Phi_p$  if and only if  $\pi$  is not one-to-one on  $\Pi_p$ . But then we again have that  $[\Pi, \Pi]$  intersects  $\Pi_p$ .
- (f) Finally, if  $|\Pi_p| > |\Phi_p|$ , then  $\Pi_p$  cannot map isomorphically onto  $\Phi_p$ , so we are done by the preceding paragraph.  $\square$

We conclude by analyzing  $\Sigma_{\Pi_p}(\mathbf{h})$ . Note that the results below that pertain only to  $\Sigma_{\Pi_p}(\mathbf{h})$  are applicable in general to all skew-primitive  $h_i$ 's, by Theorem 5.1 above.

**Theorem 7.3.** *Setup as above. Suppose moreover that  $\Pi_p \cong (\mathbb{Z}/p\mathbb{Z})^k$ .*

- (1)  $\gamma(g_i) = 1$  for all  $i$  and  $\gamma \in \Pi_p$ . In particular,  $g_i, \mathbf{g} \in G_{\Pi_p}(H)$ .
- (2) If  $k > n$ , then  $\Sigma_{\Pi}(\mathbf{h}) = \Sigma_{\Pi_p}(\mathbf{h}) = 0$ .
- (3) If  $k = n$ , then  $\Sigma_{\Pi_p}(\mathbf{h}) = \binom{p}{2}^k \cdot \text{perm}(A)$ , where  $A$  is the matrix given by  $a_{ij} = \gamma_j(h_i)$ , the  $\gamma_j$ 's form a  $\mathbb{Z}/p\mathbb{Z}$ -basis of  $\Pi_p$ , and  $\text{perm}$  is the matrix permanent:

$$\text{perm}(A_{n \times n}) = \sum_{\sigma \in S_n} \prod_{i=1}^n a_{i, \sigma(i)}$$

In particular,  $\Sigma_{\Pi_p}(\mathbf{h}) = 0$  unless  $p = 2$ , in which case we get  $\Sigma_{\Pi}(\mathbf{h}) = \Sigma_{\Pi_p}(\mathbf{h}) = \det A$ .

- (4) If  $\Sigma_{\Pi_p}(\mathbf{h}) \neq 0$ , then  $(p-1)|n$  and  $0 < k \leq n/(p-1)$ , and then  $\Sigma_{\Pi_p}(\mathbf{h})$  can take any value  $r \in R$ . (If  $k = n$  and  $p = 2$ , then  $r \neq 0$ .)

**Remark 7.3.** This result is independent of the chosen  $p$ -Sylow subgroup  $\Pi_p$ , as well as the choices of generators  $\gamma_j$ . It generalizes Proposition 3.1 above, in the special case  $p = 2$ .

The rest of this section is devoted to proving the above result. First, suppose we have a subgroup  $\Pi'_p \cong (\mathbb{Z}/p\mathbb{Z})^k$  of  $\Pi$  (so  $\Pi'_p \subset \Pi_p$  in general). Choose a set of coset representatives  $\mathcal{B}$  for  $\Pi'_p$  in  $\Pi$ , and write

$$\Sigma_{\Pi}(\mathbf{h}) = \sum_{\beta \in \mathcal{B}, \gamma \in \Pi'_p} (\gamma * \beta)(\mathbf{h}) = \sum_{\beta \in \mathcal{B}, \gamma \in \Pi'_p} \prod_{i=1}^n (\gamma * \beta)(h_i)$$

Recall that every element of  $\Pi'_p$  is  $\gamma_{a,I} := \sum_{j \in I} a_j \gamma_j$  (with  $\gamma_i$  as above), for some subset  $I$  of  $[k] := \{1, 2, \dots, k\}$ , and some  $|I|$ -tuple  $a = (a_j)_{j \in I}$  of elements of  $(\mathbb{Z}/p\mathbb{Z})^\times$ . Recall, moreover, that we had previously defined  $g_I, g'_I, h_I$  for  $I \subset [n]$ .

We also need the following lemma, that is proved using Proposition 4.2.

**Lemma 7.2.** *If  $h \in H_{g,g'}$  is pseudo-primitive with respect to  $\Pi$ , and given  $\beta \in \mathcal{B}, \gamma_{a,I} \in \Pi_p$  as above, we have*

$$(\beta * \gamma_{a,I})(h) = \prod_{j=1}^k \gamma_j(g)^{a_j} \cdot \left[ \beta(h) + \beta(g) \sum_{j=1}^k a_j \gamma_j(g^{-1}h) \right]$$

The key observation now, is that the only “monomials” that occur in the product  $\prod_{i=1}^n (\beta * \gamma_{a,I})(h_i)$  are of the form  $\beta(\mathbf{g}g_{I_0}^{-1}h_{I_0}) \cdot \prod_{j \in I} \gamma_j(h_{I_j})$ , where  $\coprod_j I_j \coprod I_0 = [n]$ , and  $I_j \subset I$  for all  $j$ . The coefficient of such a monomial in this particular summand, is  $\prod_{j \in I} a_j^{|I_j|} \gamma_j(\mathbf{g})^{a_j}$  by the lemma above. Moreover, every such monomial occurs at most once inside each  $(\beta * \gamma_{a,I})(\mathbf{h})$ .

The crucial fact that proves Theorem 7.3 above, is the following

**Key claim.** The coefficient of  $\beta(\mathbf{g}g_{I_0}^{-1}h_{I_0}) \prod_{j \in I} \gamma_j(h_{I_j})$  in  $\sum_{\gamma \in \Pi_p} (\beta * \gamma)(\mathbf{h})$ , equals  $p^{k-|I|} \prod_{j \in I} \varphi(|I_j|)$  (where  $\varphi(0) := 0$ ).

*Proof of the key claim.* (Note that  $\varphi(f)$  was defined in Lemma 7.1.) A monomial of the desired form occurs in precisely those  $(a', I')$ -summands, so that  $I' \supset I$ . Moreover, all such summands can be split up into a disjoint union over all  $a \in ((\mathbb{Z}/p\mathbb{Z})^\times)^{|I|}$ , with each disjoint piece containing all  $(a', I')$  so that  $I' \supset I$  and the  $I$ -component of  $a'$  is  $a$ .

Such a piece contains exactly  $p^{k-|I|}$  elements (and hence exactly that number of copies of the monomial with this selfsame coefficient). Each of these “extra”  $[k] \setminus I$  factors contributes a  $\beta(h_i)$ , which gives  $\beta(h_{I_0})$ .

Moreover, there is one contribution for each  $a \in ((\mathbb{Z}/p\mathbb{Z})^\times)^{|I|}$ , and it is  $\prod_{j \in I} a_j^{|I_j|} \gamma_j(\mathbf{g})^{a_j} \cdot \beta(\mathbf{g}g_{I_0}^{-1})$ , since the argument for the  $\beta$ -factor here is precisely  $\prod_{j=1}^k g_{I_j}$ . Moreover,  $\gamma_j(\mathbf{g}) = 1$  by Theorem 5.1 above.

Summing over all possible tuples  $a \in ((\mathbb{Z}/p\mathbb{Z})^\times)^{|I|}$ , we get the coefficient (apart from the  $\beta$ -part) to be

$$p^{k-|I|} \sum_a \prod_{j \in I} a_j^{|I_j|} = p^{k-|I|} \prod_{j \in I} \sum_{a_j=1}^{p-1} a_j^{|I_j|}$$

and this equals  $p^{k-|I|} \prod_{j \in I} \varphi(|I_j|)$  as desired, because the only problem may occur when some  $|I_j| = 0$ . But then  $|I| < k$ , so

$$p^{k-|I|} \sum_{a_j=1}^{p-1} a_j^0 = 0 \sum_{a_j=1}^{p-1} a_j^0 = 0 \sum_{a_j=1}^p a_j^0 = p^{k-|I|} \varphi(0)$$

□

*Proof of Theorem 7.3.*

(1) This is from Theorem 5.1 above.

We now set  $\Pi_p = \Pi'_p$ . We first note from the key claim that if  $I_0$  is nonempty, or any  $I_j$  is empty, then the coefficient of that particular monomial vanishes - because  $\text{char}(R) = p$  and  $\varphi(0) = 0$ .

- (2) Suppose  $k > n$ . Then at least one  $I_j$  must be empty in every monomial above, by the Pigeonhole Principle, and we are done.
- (3) If  $k = n$ , then the only monomials that have a nonzero contribution to the sum  $\Sigma_{\Pi'_p}(\mathbf{h})$  must correspond to empty  $I_0$  and singleton  $I_j$ 's (since  $\coprod_{j=1}^k I_j = [n] = [k]$ ). In other words, we have  $\sigma \in S_n : j \mapsto i_j \forall j$ . Moreover, the coefficient of such a monomial is  $p^0 \prod_{j=1}^n \varphi(1)$ , and these monomials all add up to give the matrix permanent, as claimed. The rest of the statements are now easy to see.
- (4) In this part, we are only concerned with  $\Sigma_{\Pi_p}(\mathbf{h})$ , so that  $\beta$  does not contribute here either (so  $I_0 = \emptyset$  and  $[n] = \coprod_{j \in I} I_j$ ).

From the key claim and Lemma 7.1 above, we see that if some monomial has a nonzero contribution, then  $(p-1)$  divides  $|I_j|$  for all  $j$ , and  $I = [k]$ . In particular,  $(p-1)$  divides  $\sum_{j \in I} |I_j| = n$ , and

$$n = \sum_{j \in I} |I_j| = \sum_{j=1}^k |I_j| \geq \sum_{j=1}^k (p-1) = k(p-1)$$

whence  $k \leq n/(p-1)$ . Moreover,  $\Sigma_{\Pi_p}(\mathbf{h}) = \varepsilon(\mathbf{h}) = 0$  if  $k = 0$ .

It remains to present, for each  $0 < k \leq n/(p-1)$  and (nonzero)  $r \in R$ , an example of  $(H, \Pi = \Pi_p)$ , so that  $\Sigma_{\Pi}(\mathbf{h}) = \Sigma_{\Pi_p}(\mathbf{h}) = r$ . We analyze this example in the next section. □

## 8. EXAMPLE 10: LIE ALGEBRAS

Suppose  $H = \mathfrak{U}\mathfrak{g}$  for some Lie algebra  $\mathfrak{g}$  (say over  $\mathbb{C}$ ). Then any weight  $\mu \in \Gamma$  kills  $[\mathfrak{g}, \mathfrak{g}]$ , hence belongs to  $(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$ . Let us denote  $\mathfrak{g}_{ab} := \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ . Conversely, any element of the set above, is a weight of  $H$ , using multiplicativity and evaluating it at the projection down to the quotient  $\mathfrak{g}_{ab}$ . Thus,  $\Gamma$  is the dual space (under addition) of the abelianization  $\mathfrak{g}_{ab}$  of  $\mathfrak{g}$ . Hence we now examine what happens in the case of an ( $R$ -free) abelian Lie algebra  $\mathfrak{h}$ .

In this case, we have the free  $R$ -module  $\mathfrak{h} = \oplus_i R h_i$  with the trivial Lie bracket, and  $H = \mathfrak{U}\mathfrak{h} = \text{Sym}(\mathfrak{h})$ . Thus,  $H$  inherits the usual Hopf algebra structure now (i.e.,  $\Delta(h_i) = 1 \otimes h_i + h_i \otimes 1$ ,  $S(h_i) = -h_i$ ,  $\varepsilon(h_i) = 0 \forall i$ ).

First,  $(\Gamma, *) = (\mathfrak{h}^*, +)$ . By Proposition 5.4, if  $\text{char}(R) \nmid |\Pi|$ , then  $\Sigma_{\Pi}(\mathbf{h}) = 0$  for all  $\mathbf{h} \in H$ . Thus, the only case left to consider is when  $\text{char}(R) = p > 0$ . But then  $(\mathfrak{h}^*, +)$  is a  $\mathbb{Z}/p\mathbb{Z}$ -vector space, so every finite subgroup is  $\Pi = \Pi_p \cong (\mathbb{Z}/p\mathbb{Z})^k$  for some  $k$ . Moreover, Theorem 3.1 and (the last part of) Theorem 7.3 provide more results in this case.

We therefore conclude the example (and the proof of the theorem above) by analyzing the computation of  $\Sigma_{\Pi}(\mathbf{h})$  for  $\mathbf{h} = h_1 \dots h_n$ . For any (nonzero)

$r \in R$ , we produce such a finite subgroup  $\Pi = \Pi_p \cong (\mathbb{Z}/p\mathbb{Z})^k$ , so that  $0 < k \leq n/(p-1)$  and  $\Sigma_{\Pi}(\mathbf{h}) = r$ .

*Construction:* Given  $k$ , partition  $[n]$  into  $k$  disjoint nonempty subsets  $[n] = \coprod_{j=1}^k I_j$ , reordered so that  $I_1 = \{1, \dots, n - (k-1)(p-1)\}$ , and so that  $|I_j| = p-1$  for all  $j > 1$ . For each  $1 \leq j \leq n$ , define  $\gamma_j \in \mathfrak{h}^* = \Gamma$  as follows:  $\gamma_1(h_1) = r$ ,  $\gamma_j(h_i) = 1$  if  $i \neq 1 \in I_j$ , and  $\gamma_j(h_i) = 0$  otherwise. (One verifies that the  $\gamma_i$ 's thus defined are indeed linearly independent over  $k(R)$ , hence over  $\mathbb{Z}/p\mathbb{Z}$  as well, but for this, we do need that  $r \neq 0$  if  $n = k, p = 2$ .) Thus for any  $K \subset [k]$ ,  $\gamma_K(h_i) := \sum_{j \in K} \gamma_j(h_i)$  vanishes unless  $i \in \cup_{j \in K} I_j$ .

Now evaluate  $\Sigma_{\Pi}(\mathbf{h}) = \sum_{K \subset [k], a} \prod_{i=1}^n \gamma_{a,K}(h_i)$ , where we define  $\Pi := \sum_{i=1}^n \mathbb{Z}\gamma_i = \bigoplus_{i=1}^n (\mathbb{Z}/p\mathbb{Z})\gamma_i$ . By the key claim in the previous section, the only monomials  $\prod_{j \in I} \gamma_j(h_{I'_j})$  that do not vanish are for  $|I| = k$ , and with  $(p-1)$  divides  $|I'_j|$  for all  $j$ . Moreover,  $\gamma_j(h_i)$  is zero except when  $i \in I_j$ , so there is only one type of monomial remaining:  $\prod_{j \in I} \gamma_j(h_{I_j})$ . (Note that this satisfies the earlier condition:  $(p-1)$  divides  $|I_j|$  for all  $j$ .)

Moreover, by the key claim in the preceding section, the coefficient of this monomial, which itself equals  $r \cdot \prod_{i=2}^n 1 = r$ , is  $\prod_{j=1}^k \varphi(|I_j|)$ , and by Fermat's Little Theorem,  $\varphi(|I_j|) = p-1 = -1 \forall j$  (in characteristic  $p$ ). We conclude that  $\Sigma_{\Pi}(\mathbf{h}) = \Sigma_{\Pi_p}(\mathbf{h}) = (-1)^k r$ , whence we are done (start with  $r' = (-1)^k r$  to get  $r$ ).

## 9. EXAMPLE 11: DEGENERATE AFFINE HECKE ALGEBRAS OF REDUCTIVE TYPE WITH TRIVIAL PARAMETER

In this section, we apply the general theory above, to a special case, wherein a finite group acts on a vector space (or free  $R$ -module in our case), with the group and the module corresponding to the Weyl group and the Cartan subalgebra (actually, its dual space) respectively, of a reductive Lie algebra. We use the  $\mathbb{Z}$ -basis of simple roots (and any  $\mathbb{Z}$ -basis for the center), to try and compute the value of  $\Sigma_{\Pi}(\mathbf{h})$ .

**9.1. Hopf algebras acting on vector spaces.** We consider special cases of the following class of Hopf algebras. Suppose that a cocommutative  $R$ -Hopf algebra  $H$  acts on a free  $R$ -module  $V$ ; denote the action by  $h(v)$  for  $h \in H, v \in V$ . Then  $H$  also acts on  $V^*$  by:  $\langle h(\lambda), v \rangle := \langle \lambda, S(h)(v) \rangle$ .

We now consider the  $R$ -algebra  $A$  generated by the sets  $H$  and  $V$ , with obvious relations in  $H$ , and the extra relations  $vv' = v'v$ ,  $\sum h_{(1)}vS(h_{(2)}) =: \text{ad } h(v) = h(v)$  for all  $h \in H$  and  $v, v' \in V$ . We remark that the relation  $\text{ad } h(v) = h(v)$  can be rephrased, as the following lemma shows:

**Lemma 9.1.** *Suppose some  $R$ -Hopf algebra  $H$  acts on a free  $R$ -module  $V$ , and an  $R$ -algebra  $B$  contains  $H, V$ . Then the following relations are equivalent (in  $B$ ) for all  $v \in V$ :*

$$(1) \sum h_{(1)}vS(h_{(2)}) = h(v) \text{ for all } h \in H.$$

(2)  $hv = \sum h_{(1)}(v)h_{(2)}$  for all  $h \in H$ .

If  $H$  is cocommutative, then both of these are also equivalent to:

(3)  $vh = \sum h_{(1)}S(h_{(2)})(v)$  for all  $h \in H$ .

Moreover, if this holds, then any unital subalgebra  $M$  of  $B$  that is also an  $H$ -submodule (via  $\text{ad}$ ), is an  $H$ -(Hopf-)module algebra under the action

$$h(m) := \text{ad } h(m) = \sum h_{(1)}mS(h_{(2)}) \quad \forall h \in H, m \in M$$

(The proof is straightforward.) For instance, we can take  $M = B$  or  $H$  - or in the above example of  $A$ , we can consider  $M = \text{Sym}_R V$ .

It is straightforward (but perhaps tedious) to check that  $A$  is a Hopf algebra with the usual operations: on  $H$ , they restrict to the Hopf algebra structure of  $H$ , and  $V$  consists of primitive elements.

By the above lemma, if  $H$  is  $R$ -free, then the ring  $A$  is an  $R$ -free  $R$ -Hopf algebra, with  $R$ -basis given by  $\{h \cdot m\}$ , where  $h \in H$  and  $m$  run respectively over some  $R$ -basis of  $H$ , and all (monomial) words (including the empty word) with alphabet given by an  $R$ -basis of  $V$ . It has the subalgebras  $H$  and  $\text{Sym}_R(V)$ , and is called the *smash product*  $H \rtimes \text{Sym}_R V$  of  $H$  and  $\text{Sym}_R V$ .

We now determine the weights of  $A$ . Denote by  $\Gamma_H$  the group of weights of  $H$  (under convolution). We can now use Proposition 2.2 to prove:

**Proposition 9.1.** *The weights  $\Gamma = \Gamma_A$  of  $A$  form a group, which is the Cartesian product  $\Gamma_H \times V_\varepsilon^*$ , with convolution given by*

$$(\nu_1, \lambda_1) * (\nu_2, \lambda_2) = (\nu_1\nu_2, \lambda_1 + \lambda_2) = (\nu_1 *_H \nu_2, \lambda_1 *_V \lambda_2)$$

for  $\nu_i \in \Gamma_H$ ,  $\lambda_i \in V_\varepsilon^*$ . (Here,  $V_\varepsilon^*$  is the  $\varepsilon$ -weight space of the  $H$ -module  $V^*$ .)

**9.2. Degenerate affine Hecke algebras.** Since we work over any commutative unital integral domain  $R$ , hence we can generate examples over all  $R$ , if we have a lattice in  $V$  that is fixed by  $H$ , and we then look at the  $R$ -span. We now specialize to the case when  $H = RW$  is the group ring of a Weyl group acting on a Cartan subalgebra of the corresponding semisimple Lie algebra. Then one uses the root lattice  $Q$  inside  $V = \mathfrak{h}^*$ .

We work in slightly greater generality. Given a finite-dimensional reductive complex Lie algebra  $\mathfrak{g}$ , let  $W$  be its Weyl group and  $\mathfrak{h}$  a fixed chosen Cartan subalgebra. Thus  $\mathfrak{h} = \bigoplus_{i \geq 0} \mathfrak{h}_i$ , where for  $i > 0$ ,  $\mathfrak{h}_i$  corresponds to a simple component (ideal) of  $\mathfrak{g}$ , with corresponding base of simple roots  $\Delta_i$  and Weyl group  $W_i$ , say; and  $\mathfrak{h}_0$  is the central ideal in  $\mathfrak{g}$ .

We define  $Q_i = \bigoplus_{\alpha \in \Delta_i} \mathbb{Z}\alpha$ , the root lattice inside  $\mathfrak{h}_i^*$ , and choose and fix some  $\mathbb{Z}$ -lattice  $Q_0$  inside  $\mathfrak{h}_0^*$ . Now replace  $\mathfrak{h}_i^*$  by  $V_i = \mathfrak{h}_i^* := R \otimes_{\mathbb{Z}} Q_i$ , and  $\mathfrak{h}_i$  by the  $R$ -dual of  $\mathfrak{h}_i^*$ , for all  $i \geq 0$ . Thus, for the entire Lie algebra, we have  $\Delta = \coprod_{i > 0} \Delta_i$  and  $W = \times_{i > 0} W_i$ .

We now define  $V = \bigoplus_{i \geq 0} V_i$ , whence the previous subsection applies and we can form the algebra  $A = RW \rtimes \text{Sym}_R V$ . This is the *degenerate affine Hecke algebra with trivial parameter* (the parameter is trivial since

$wv - w(v)w$  is always zero), of reductive type. This is a special case of [4, Definition 1.1], where we set  $\eta = 0$ .

Before we try to address the general case, note that there are two types of  $\mathfrak{h}_i$ 's in here: ones corresponding to simple Lie algebras, which we address first, and the ‘‘central part’’, which is fixed by  $W$  (hence so is  $\mathfrak{h}_0^*$ ).

**9.3. The simple case.** We first consider the case:  $V = \mathfrak{h}^* = R \otimes_{\mathbb{Z}} Q$  for a simple Lie algebra. Thus  $\Delta$  is irreducible, and given  $A = RW \ltimes \text{Sym}_R(\mathfrak{h}^*)$ , we have  $\Gamma = \Gamma_W \times \mathfrak{h}^W$  (because the condition in Proposition 9.1 above translates to:  $w(\gamma) = \varepsilon(w)\gamma = \gamma$  for all  $w \in W, \gamma \in \Gamma$ ). Here,  $\Gamma_W = \Gamma_{RW}$ .

We now state our main result, using the convention that all roots in the simply laced cases (types  $A, D, E$ ) are short. The result helps compute  $\Sigma_{\Pi}$  at any element of the  $R$ -basis  $\{g \cdot m\}$  mentioned in an earlier subsection.

**Theorem 9.1.** *Setup as above.*

- (1) *If  $\text{char}(R) \neq 2$ , or  $W$  is of type  $G_2$ , or  $W$  has more than one short simple root, then every weight acts as  $\varepsilon = 0$  on  $\mathfrak{h}^*$ . In particular,  $\Sigma_{\Pi} = 0$  on  $\text{Sym}_R(\mathfrak{h}^*)$ .*
- (2) *If  $\text{char}(R) = 2$ , then every weight acts as  $\varepsilon$  on  $W$ . Now suppose also that  $W$  is not of type  $G_2$ , and has only one short simple root  $\alpha_s$ , say. If  $\Pi$  has an element of order 4, or  $h_i$  has no ‘‘ $\alpha_s$ -contribution’’ (i.e.,  $h_i \in \bigoplus_{\alpha_s \neq \alpha \in \Delta} R \cdot \alpha$ ) for some  $i$ , then  $\Sigma_{\Pi}(\mathbf{h}) = 0$ .*
- (3) *If this does not happen, i.e.,  $\Pi = (\mathbb{Z}/2\mathbb{Z})^k$  for some  $k$ , and the hypotheses of the previous part hold, then*

$$\Sigma_{\Pi}(\alpha_s^n) = \sum_{\substack{l_i > 0 \forall i \\ l_1 + \dots + l_k = n}} \binom{n}{l_1, \dots, l_k} \prod_{i=1}^k \gamma_i(\alpha_s)^{l_i}$$

where the  $\gamma_i$ 's are any set of generators for  $\Pi$ . In particular, this vanishes if  $k > r$ , where  $\sum_{j=1}^r 2^{s_j}$  is the binary expansion of  $n$ .

**Remark 9.1.**

- (1) **Warning.** Do not confuse the  $h_i$ 's here with elements of  $\mathfrak{h}$ ; indeed,  $h_i \in A$ , so they really are in  $\mathfrak{h}^*$ , and I apologize for this notation!
- (2) The coefficient above is just the multinomial coefficient  $n! / (\prod_i l_i!)$ , which we also denote by  $\binom{n}{l_1, \dots, l_{k-1}}$ , just as  $\binom{n}{k, n-k} = \binom{n}{k}$ . The last line in the theorem follows because this coefficient is odd if and only if ( $r, s_j$  as above) we can partition  $\{2^{s_j} : j\}$  into  $k$  nonempty subsets, and the  $l_i$ 's are precisely the sums of the elements in the subsets. (This fails, for instance, if some two  $l_i$ 's are equal, or  $k > r$ .) This fact, in turn, follows (inductively) from the following easy-to-prove

**Lemma 9.2.** *Suppose  $p > 0$  is prime,  $p^s \leq n < p^{s+1}$  for some  $s \geq 0$ , and  $l_k \geq l_i \forall i$ . If  $l_k < p^s$  then  $p$  divides  $\binom{n}{l_1, \dots, l_{k-1}}$ . Otherwise  $p$  divides neither or both of  $\binom{n}{l_1, \dots, l_{k-1}}$  and  $\binom{n-p^s}{l_1, \dots, l_{k-1}}$ .*

The rest of the subsection is devoted to the proof of the theorem. We once again mention a result crucial to the proof, then use it to prove the theorem, and conclude by proving the key claim.

**Key claim.** ( $\text{char}(R)$  arbitrary.) If  $W$  contains a Dynkin subgraph  $\Omega$  of type  $A_2$  or  $G_2$ , then both the simple roots in  $\Omega$  are killed by all  $\lambda \in \mathfrak{h}^W$ . If  $\Omega$  is of type  $B_2$ , then the long root in  $\Omega$  is killed by all  $\lambda$ .

*Proof modulo the key claim.* We now show the theorem.

- (1) First suppose that  $\text{char}(R) \neq 2$ . If  $\lambda \in \mathfrak{h}^W$ , then  $\lambda(\alpha) = \lambda(s_\alpha(\alpha)) = -\lambda(\alpha)$ , whence  $\lambda(\alpha) = 0$  for all  $\alpha \in \Delta$ , and  $\mathfrak{h}^W = 0$ .

For the other claims, we use the classification of simple Lie algebras in terms of Dynkin diagrams, as mentioned in [8, Chapter 3]. To show that a weight  $\lambda$  kills all of  $\mathfrak{h}^*$ , it suffices to show that  $\lambda(\alpha) = 0 \forall \alpha \in \Delta$ , i.e., that it kills each simple root, or node of the corresponding Dynkin diagram.

If the Dynkin diagram of a Lie algebra has (a sub-diagram of) type  $A_2$  or  $G_2$ , then both nodes of that diagram (or both  $\alpha_i$ 's) are killed by all weights  $\lambda \in \Gamma$ , by the key claim above. This automatically eliminates all diagrams of type  $A_n$  for  $n > 1$ , as well as all  $D, E, F, G$ -type diagrams, leaving only type  $A_1$  among these.

Moreover, for types  $B, C$ , at most one simple root (the “last” one) is not killed by all  $\lambda$ 's. If this root is long, then it is also killed by the key claim above (as a part of a  $B_2$ ), and we are done.

- (2) First,  $\lambda(s_\alpha^2) = \lambda(s_\alpha)^2 = 1$ , whence  $\lambda(s_\alpha) = \pm 1 = 1 \forall \alpha \in \Delta$ , if  $\text{char}(R) = 2$ . This implies that  $\lambda(w) = 1 = \varepsilon(w)$  for all  $w \in W, \lambda \in \Gamma$ . Next, Theorem 7.2 above tells us that if  $\Pi$  has an element of order 4, then  $\Sigma_\Pi(\mathbf{h}) = 0$ . Finally, if some  $h_i$  has no “ $\alpha_s$ -contribution”, then it is killed by all  $\lambda$ , by the previous part, so  $\lambda(\mathbf{h}) = 0 \forall \lambda \in \Gamma$ .

- (3) As we remarked after Theorem 7.3,  $\Pi = (\mathbb{Z}/2\mathbb{Z})^k$  in characteristic 2, if  $\Pi$  does not have an element of order 4. (Reason:  $\Gamma = \{\varepsilon\} \times \mathfrak{h}^W \cong (\mathfrak{h}^W, +)$  is a free  $R$ -module by the previous part, and  $2\Gamma = 0$ .)

We now perform the computation. For this, suppose that  $h_i - c_i \alpha_s$  is in the  $R$ -span of  $\{\alpha \in \Delta : \alpha \neq \alpha_s\}$  (note that in the case of  $A_1$ , the condition  $h_i \in R \cdot \alpha_s$  is automatic). Then  $\Sigma_\Pi(\mathbf{h}) = (\prod_i c_i) \cdot \Sigma_\Pi(\alpha_s^n)$ , so it suffices to compute  $\Sigma_\Pi(\alpha_s^n)$ .

If  $\{\gamma_j\}$  is any set of generators (or  $\mathbb{Z}/2\mathbb{Z}$ -basis) for  $\Pi$ , then the desired equation actually holds if we sum over all *nonnegative* tuples  $l_i$ , that add up to  $k$ . Thus, the proof is similar to that of the key claim used to prove Theorem 7.3 above; we simply need to note that if  $I \subsetneq [k]$ , then every  $\prod_{j \in I} \gamma_j(\alpha_s^{|I_j|})$  occurs with an even coefficient.  $\square$

Finally, we prove the key claim.

*Proof of the key claim.* It helps to look at the pictures of these rank 2 root systems (drawn in [8, Chapter 3]). We use the  $W$ -invariance of  $\lambda|_V \forall \lambda \in \Gamma$ .

Consider the system  $A_2$ , with simple roots  $\alpha, \beta$ . Given  $\lambda \in \Gamma$ , we have  $\lambda(\alpha) = \lambda(\beta) = \lambda(\alpha + \beta)$ , whence  $\lambda(\alpha) = \lambda(\beta) = 0$ .

The root system  $G_2$  has two subsystems of type  $A_2$ , whence each  $\lambda$  must kill both subsystems.

Now consider  $B_2$ , with long root  $\alpha$  and short root  $\beta$ . Clearly,  $\beta + \alpha$  is another short root, whence  $\lambda(\beta + \alpha) = \lambda(\beta)$ , and we are done.  $\square$

**9.4. The reductive case.** We conclude by mentioning what happens in the reductive case. We need the results proved in the simple case above. Recall also that we had set our notation for this situation, when we defined degenerate affine Hecke algebras with trivial parameter earlier. We now use this notation freely, without recalling it from there.

Let  $V'$  be the direct sum of  $V_0$  and the  $R$ -span of all the unique short simple roots  $\alpha_i$ , *short* inside any of the simple components  $V_i = \mathfrak{h}_i^*$  of the “correct” type (not  $G_2$ ). Let the other simple roots in  $\Delta$  span the  $R$ -submodule  $V''$ . Then  $V = V' \oplus V''$ , and each  $\lambda \in \Gamma$  kills  $V''$ . We now have two cases.

**Case 1.**  $\text{char}(R) \neq 2$ . Then  $\lambda$  in fact kills all  $\alpha \in \Delta$ , because  $\lambda(\alpha) = \lambda(s_\alpha(\alpha)) = -\lambda(\alpha)$ . This means that we are left with  $V_0$ , i.e., if for all  $i$ , we have  $h_i - v_{0,i} \in \bigoplus_{j>0} V_j$  for some  $v_{0,i} \in V_0$ , then  $\Sigma_\Pi(\mathbf{h}) = \Sigma_\Pi(\prod_i v_{0,i})$ .

Next, recall that  $\Sigma_\Pi = \Gamma_W \times (V^*)^W$ , so we are reduced to the case of every  $\lambda$  being represented (on  $V_0$ ) by some element of  $V_0^* = (V_0^*)^W$ . We conclude this case by noting that we had some (partial) results on how to compute this, in the previous section.

**Case 2.**  $\text{char}(R) = 2$ . Then  $\lambda(w) = 1$  for all  $w, \lambda$ , as seen above. Moreover, we are left only to consider the case of all  $h_i \in V'$ . Now,  $\Gamma = \varepsilon_W \times (V')^*$ , whence any finite subgroup  $\Pi = (\mathbb{Z}/2\mathbb{Z})^k$  for some  $k$  (since it too is a  $\mathbb{Z}/2\mathbb{Z}$ -vector space). In this situation, Theorems 7.2 and 7.3 (and 9.1 as well) give us some information on how to compute  $\Sigma_\Pi(\mathbf{h})$ .

## 10. AN EXAMPLE THAT ATTAINS ANY VALUE

We conclude with examples where  $\Sigma_\Pi(\mathbf{h})$  can take any value in  $R$ , if the  $h_i$ 's are merely skew-primitive.

**Example 12** (A skew-primitively generated algebra). By Proposition 5.4 above, if all  $h_i$ 's are pseudo-primitive, then  $\Sigma_\Pi(\mathbf{h}) = 0$  if  $\text{char}(R) \nmid |\Pi|$  - whereas if  $\text{char}(R)$  divides  $|\Pi|$ , then we have analyzed the situation in Section 7 above.

We may ask if such results hold in general, i.e., for products of skew-primitive elements. (Note by Theorem 5.1 that we need:  $\text{char}(R) \nmid |\Pi|$ .)

For the example that we now mention (for groups  $\Pi$  of *even* order), we need to **assume** the following:

- (1)  $\text{char}(R) > 2^n$  and  $\exp(\Pi)$ , or  $\text{char}(R) = 0$  and  $R \supset \mathbb{Q}$ ; and

- (2) If  $d = \exp(\Pi)$  is the exponent, then  $d$  is even, and there exists a primitive  $d$ th root of unity in  $R$ , say  $z$ .

Beyond this, given  $n, \Pi$  (of even order), and  $r \in R$ , we will produce the desired Hopf algebra  $\mathcal{H}$ , a group of weights  $\Pi \subset \Gamma_{\mathcal{H}}$ , and skew-primitive  $h_1, \dots, h_n \in H$ , such that  $\Sigma_{\Pi}(\mathbf{h}) = r \in R$ .

Given  $\Pi$ , use the Structure Theorem for finite Abelian groups, to write:  $\Pi = \bigoplus_{i=1}^k (\mathbb{Z}/d_i\mathbb{Z})$ , with  $d_1|d_2|\dots|d_k = \exp(\Pi)$ . Then  $d_k$  is even, since  $\Pi$  has even order. Now define  $\mathcal{H}$  to be the commutative  $R$ -algebra freely generated as:  $\mathcal{H} = R[R^n] \otimes R[\mathbb{Z}^k]$ . In other words,  $R$  is generated by  $h_1, \dots, h_n, g_1^{\pm 1}, \dots, g_k^{\pm 1}$ , with the relation that they all commute (and that the  $g_i$ 's are invertible).

Now define the  $g_i$ 's to be grouplike and  $\Delta(h_j) = g_k \otimes h_j + h_j \otimes 1$ . Also define (for all  $i, j$ ):

$$\varepsilon(g_i) = 1, S(g_i) = g_i^{-1}, \varepsilon(h_j) = 0, S(h_j) = -g_k^{-1}h_j$$

Since  $\mathcal{H}$  is freely generated, the set of weights of  $\mathcal{H}$  is  $R^n \times (R^\times)^k$ . Since (it can be checked that)  $\mathcal{H}$  is also a Hopf algebra, the group operation is:

$$\begin{aligned} & (a_1, \dots, a_n, z_1, \dots, z_k) * (a'_1, \dots, a'_n, z'_1, \dots, z'_k) \\ &= (a_1 + z_k a'_1, a_2 + z_k a'_2, \dots, a_n + z_k a'_n, z_1 z'_1, \dots, z_n z'_n) \end{aligned}$$

We now produce the desired example. Define  $\gamma_j \in \Gamma_{\mathcal{H}}$  on generators by:  $\gamma_j(g_i) = z^{\delta_{ij}d_k/d_j}$ , and  $\gamma_j(h_i) = 0$  unless  $j = k$ . Moreover,  $\gamma_k(h_i) = 1$  for  $i < k$ , and  $\gamma_k(h_k) = (1 - z)^n |\Pi|^{-1} r'$  for some  $r' \in R$  (which we will define later, and which depends on  $n$ ).

It is now easy to check that each  $\gamma_j$  is of order  $d_j$ , and the  $\gamma_j$ 's generate a subgroup of  $\Gamma_{\mathcal{H}}$  isomorphic to  $\Pi$ . Moreover,  $\gamma_1, \dots, \gamma_{k-1}$  all kill  $h_1, h_2, \dots, h_n$ . Adopting the notation of Proposition 4.1,  $\gamma_j \in \Gamma_{h_i}$  for  $j < k$  and all  $i$ . Since  $\Gamma_{h_i}$  is a subgroup of  $\Gamma_{\mathcal{H}}$ , hence  $\Pi_1 := \langle \gamma_1, \dots, \gamma_{k-1} \rangle \subset \Gamma_{h_i}$  for all  $i$ ; by Proposition 4.1,  $\Pi_1 \subset \Gamma_{\mathbf{h}}$ .

Now use Lemma 4.2 (noting that  $\Pi_1$  is normal in the abelian group  $\Pi$ ); then  $\Sigma_{\Pi}(\mathbf{h}) = |\Pi/\Pi_1| \Sigma_{\Pi_1}(\mathbf{h})$ . We now use Theorem 6.2:  $f_i = \frac{\gamma_k(h_i)}{\gamma_k(g_k)^{-1}}$ , so

$$(-1)^n \prod_i f_i = (-1)^n \frac{(1-z)^n |\Pi|^{-1} r'}{\prod_{i=1}^n (z-1)} = \frac{r'}{|\Pi|}$$

Moreover,  $S = \{I \subset \{1, 2, \dots, n\} : g_k^{|I|} \text{ is fixed by } \Pi_1\}$ , i.e., all subsets  $I$  such that  $d_k ||I|$ . Since  $d_k$  is even, this means that  $(-1)^{|I|} = 1 \forall I \in S$ , whence

$$\Sigma_{\Pi}(\mathbf{h}) = |\Pi/\Pi_1| \Sigma_{\Pi_1}(\mathbf{h}) = \frac{|\Pi|}{|\Pi_1|} \cdot |\Pi_1| \cdot \frac{r'}{|\Pi|} \cdot \sum_{m \geq 0} \binom{n}{md_k} = r' \sum_{m \geq 0} \binom{n}{md_k}$$

By assumption (on  $R$ ), the summation is a unit in  $R$ , so we can choose  $r'$  suitably, to get any  $r \in R$  as our answer.  $\square$

Also note (e.g., by [11, Exercise 38, §1.2.6]), that the summation equals  $\frac{1}{d_k} \sum_{l=0}^{d_k-1} (1+z^l)^n$ .

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