

# BLOCK DECOMPOSITION FOR CATEGORY $\mathcal{O}$ OVER ALGEBRAS OF CRAWLEY-BOEVEY TYPE

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## 0. References.

W. Crawley-Boevey and M.P. Holland, *Noncommutative deformations of Kleinian singularities*, Duke Math. **92** (1998), 605–635.

A. Khare, *Axiomatic Framework for the BGG Category  $\mathcal{O}$* , math.RT/0502227

## 1. A simple problem on complex numbers.

Suppose we have nonzero complex numbers  $z_i \in \mathbb{C}^*$ , so that the sequence  $\{z_1^n - z_2^n + 2z_3^n : n \in \mathbb{N}\}$  contains some complex number infinitely often. What are all solutions?

For instance,  $z_1 = \zeta z_2$ , and  $z_3, \zeta$  are roots of unity.

In general, consider the problem for  $\{\sum_{i=1}^k b_i z_i^n : n \in \mathbb{N}\}$ , where  $z_i, b_i \in \mathbb{C}^*$ .

## 2. Connection with representation theory

Given a subgroup  $\Gamma$  of the maximal torus  $\mathbb{C}^* \subset SL_2(\mathbb{C})$ , which acts on  $V = \mathbb{C}^2 = \mathbb{C}X \oplus \mathbb{C}Y$ , and  $\lambda \in \mathbb{C}\Gamma$ , define a  $\mathbb{C}$ -algebra  $\mathcal{S}^\lambda$  by

$$\mathcal{S}^\lambda := (TV \rtimes \Gamma)/(XY - YX - \lambda)$$

Then

- (*PBW Property.*)  $\mathcal{S}^\lambda \cong \mathbb{C}[Y] \otimes \mathbb{C}\Gamma \otimes \mathbb{C}[X]$  as vector spaces.
- $\Gamma$  has a character  $\alpha : \Gamma \rightarrow \mathbb{C}^*$  given by  $\alpha(\gamma) = (\gamma)_{11}$ .  
 $\alpha$  and  $\Gamma$  have the same orders.
- (*Weight spaces.*)  $\gamma X \gamma^{-1} = \alpha(\gamma)X$ ,  $\gamma Y \gamma^{-1} = \alpha^{-1}(\gamma)Y$ .  
Thus,  $X^n \in (\mathcal{S}^\lambda)_{\alpha^n}$ ,  $Y^n \in (\mathcal{S}^\lambda)_{\alpha^{-n}}$ .

### 3. Category $\mathcal{O}$

**Assumption.** Henceforth,  $\Gamma$  is infinite. We write  
 $\lambda = \sum_{\gamma} a_{\gamma} \gamma \in \mathbb{C}\Gamma$ .

**Category  $\mathcal{O}$ .** This is the (full sub)category of all finitely generated  $\mathcal{S}^\lambda$ -modules, that are  $\Gamma$ -semisimple with finite-dimensional weight spaces, and  $X$  acts locally finitely on any vector in each object.

- Simple objects and Verma modules in  $\mathcal{O}$  are in bijection with the set (group)  $G$  of *characters* of  $\Gamma$ .
- Given  $\mu \in G$ , let  $Z(\mu), V(\mu)$  denote the Verma and simple modules of highest weight  $\mu$ .  
Then  $Z(\mu) \rightarrow V(\mu) \rightarrow 0$ , and  $Z(\mu) \cong \mathbb{C}[Y]$  as free  $\mathbb{C}[Y]$ -modules.  
If  $v_\mu$  is the highest weight vector, then a weight-basis is  $\{Y^n v_\mu : n \geq 0\}$ . Thus, maps between Verma modules are unique upto scalars.

## 4. Verma modules

**Proposition 1.** *Given  $\mu, \nu \in G$ ,  $Z(\nu) \hookrightarrow Z(\mu)$  if and only if  $\nu = \alpha^{-n}\mu$  for some  $n \geq 0$ , and  $f_\lambda(\mu, \nu) = 0$ , where*

$$f_\lambda(\mu, n) := na_{\text{id}} + \sum_{\gamma \neq \text{id}} \frac{a_\gamma \mu(\gamma)}{1 - \gamma_{11}^{-1}} (1 - \gamma_{11}^{-n})$$

*Proof.* If  $\nu = \alpha^{-n}\mu$ , the vector  $Y^n v_\mu$  is maximal, so  $X(Y^n v_\mu) = 0$ . Now compute.  $\square$

**Proposition 2.** *If  $K_\mu := \{0 < n_1 < n_2 < \dots : f_\lambda(\mu, n_i) = 0\}$ , then the only submodules of  $Z(\mu)$  occur in the chain*

$$Z(\mu) \supset Z(\alpha^{-n_1}\mu) \supset Z(\alpha^{-n_2}\mu) \supset \dots$$

*If  $n \in \mathbb{N}$  is the least element in  $K_\mu$ , then  $V(\mu)$  is finite-dimensional with weight basis  $\{v_{\alpha^{-i}\mu} : 0 \leq i < n\}$ .*

*All finite-dimensional modules in  $\mathcal{O}$  are completely reducible if and only if no Verma module has length 3 or more.*

**Proposition 3.** *The following are equivalent for  $\mu, \nu \in G$ :*

- (1) *There is a nonzero module map  $\varphi : Z(\nu) \rightarrow Z(\mu)$ .*
- (2) *There is an injection  $\varphi : Z(\nu) \hookrightarrow Z(\mu)$ .*
- (3)  *$V(\nu)$  is a subquotient of  $Z(\mu)$ .*

## 5. Condition (S)

Introduce an equivalence relation among the set  $G$  of weights. We say that  $\mu \sim \nu$  if  $Z(\nu) \hookrightarrow Z(\mu)$ , and look at the closure under symmetry and transitivity.

Call this equivalence class  $S(\mu)$ . Thus  $G = \coprod_{\mu \in \Theta} S(\mu)$  for some set  $\Theta$  of representatives.

(This is similar to the central character block decomposition in the case of a semisimple Lie algebra  $\mathfrak{g}$  over  $\mathbb{C}$ .)

**Definition.** *Condition (S)* holds if every  $S(\mu)$  is finite.

(In case of  $\mathfrak{g}$ , each  $S(\mu)$  is contained inside a twisted Weyl orbit  $W \bullet \mu$ , hence is finite.)

**Theorem.** *If Condition (S) holds, then  $\mathcal{O}$  splits as a direct sum of subcategories  $\mathcal{O}(\mu)$  ( $\mu \in \Theta$ ), so that*

- *each  $\mathcal{O}(\mu)$  is a highest weight category with enough projectives and BGG Reciprocity.*
- *morphisms and extensions between different blocks are trivial.*

**Example.** If  $XY - YX = \lambda = 1$ , then every Verma module is simple,  $\mathcal{O}$  is semisimple, and Condition (S) holds.

## 6. Connection with sums of complex powers

Recall that  $\lambda = \sum_{\gamma} a_{\gamma} \gamma \in \mathcal{S}^{\lambda}$ .

**Proposition 4.** *Suppose  $\lambda - c\gamma_0 - d\text{id}$  is in the span  $S_{fin}$  of elements  $\gamma \neq \text{id}$  of finite order, with  $\gamma_0 \notin S_{fin}$ ,  $\gamma_0 \neq \text{id}$ . Then Condition (S) fails if  $c = d = 0$ , but holds otherwise.*

Now **assume** that  $a_{\text{id}} = 0$ . Then  $\lambda - \sum_i a_i \gamma_i \in S_{fin}$ , with  $\gamma_i \notin S_{fin}$ ,  $\gamma_i \neq \text{id} \forall i$ .

Then the failure of Condition (S), or the infinitude of an equivalence class  $S(\mu)$ , is equivalent to the existence of infinitely many  $n$  so that  $f_{\lambda}(\mu, n) = 0$ .

This translates back to our original problem on complex numbers, with  $z_i = \alpha(\gamma_i) = (\gamma_i)_{11}$  for all  $i$ .