DRINFELD-HECKE ALGEBRAS OVER COCOMMUTATIVE ALGEBRAS

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ABSTRACT. If A is a cocommutative algebra with coproduct, then so is the smash product algebra of a symmetric algebra $\operatorname{Sym} V$ with A, where V is an A-module.

Such smash product algebras, with A a group ring or a Lie algebra, have been deformed by Drinfel'd and more recently, Crawley-Boevey and Holland, Etingof and Ginzburg (and Gan), and others. These algebras include symplectic reflection algebras and infinitesimal Hecke algebras.

We introduce a family of deformations $\mathscr{H}_\beta$ of the smash product algebras mentioned at the beginning of the abstract, by deforming the relations $V \wedge V$. Thus $\beta : V \wedge V \to A \oplus V$; we characterize the β's for which the PBW property holds. We then analyse in detail the case where $A = NC_W$ is the nilCoxeter algebra, and $\beta : V \wedge V \to A$.

In the case where A is a cocommutative Hopf algebra, that β is A-compatible, is equivalent to some other conditions - that β is an A-module map, or the Yetter-Drinfeld condition. We examine what further conditions are needed on β to achieve a Hopf algebra structure on the deformed algebra (with V primitive). Finally, we provide a Hopf-theoretic analogue of symplectic reflections.

1. Preliminaries

1.1. Setup. We first set some notation.

- Throughout this work, R denotes a commutative unital integral domain whose quotient field $k(R)$ satisfies: $\operatorname{char} k(R) \neq 2$. (This is so that R-valued skew-symmetric bilinear forms $\beta : V \wedge V \to R$ satisfy $\beta(v, v) = 0 \ \forall v \in V$.)
- By $\dim V$ for a free R-module V, we will mean the rank of V. All R-modules (including all R-algebras) are assumed to be R-free. Unless otherwise specified, all (Hopf) algebras, modules, and bases (of free modules) are with respect to R.
- A *weight* of an R-algebra H is an R-algebra map : $H \to R$ that sends 1 to 1. Denote the set of weights of H by Γ_H. Given $\lambda \in \Gamma_H$ and an H-module M, denote the H-action by $h(m)$ for $h \in H, m \in M$, and define the λ-*weight space* to be

$$M_\lambda := \{m \in M : h(m) = \lambda(h)m \text{ for all } h \in H\}$$

Date: August 14, 2007.

2000 *Mathematics Subject Classification.* 16W30 (Primary), 16S40 (Secondary).

- Whenever we encounter a R-Hopf algebra, we will denote its operations by $\eta, \Delta, \varepsilon, S$ for the unit, comultiplication, counit, and antipode respectively. We will use Sweedler notation: $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}, \Delta^{(2)}(a) = \sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$, etc.
- Suppose a Hopf algebra A is a subalgebra of an algebra B. The *adjoint action* of A on B is

$$\operatorname{ad} a(b) := \sum a_{(1)} b S(a_{(2)}) \ \forall a \in A,\ b \in B$$

- Suppose that an (R-free) Hopf algebra A acts on a free R-module V. Then A also acts on $V^* := \operatorname{Hom}_{R-mod}(V, R)$ by: $\langle a(\lambda), v\rangle := \langle \lambda, S(a)(v)\rangle$.

Definition 1.1. Suppose A is an R-free unital associative R-algebra.

(1) An *algebra with coproduct* is A together with any algebra map $\Delta : A \to A \otimes_R A$, that satisfies
 (a) $\Delta(1) = 1 \otimes 1$.
 (b) Δ is *coassociative*; i.e. $(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta : A \to A \otimes A \otimes A$.

 Given $a \in A$, we write $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$ and $\Delta^{op}(a) = \sum a_{(2)} \otimes a_{(1)}$.
(2) An algebra with coproduct is *cocommutative* if $\Delta = \Delta^{op}$.

Remark 1.2.

(1) This definition differs from that of a *bialgebra*, which also involves the counit. Moreover, a coproduct allows us to take tensor products of A-modules. Finally, Δ allows us to define an algebra structure on H^* (and Γ_H is the submonoid of grouplike elements in H^*), via *convolution*: $(\lambda * \mu)(h) := (\lambda \otimes \mu)(\Delta(h)) = \sum \lambda(h_{(1)})\mu(h_{(2)})$.
(2) We will use Sweedler notation as usual; the context will indicate whether the algebra (with coproduct) in question is a Hopf algebra or not.
(3) Bialgebras and Hopf algebras (with the usual coproduct) are examples of algebras with coproduct.
(4) Every unital R-algebra A is an algebra with coproduct, if we define $\Delta_L(a) := a \otimes 1$ or $\Delta_R(a) := 1 \otimes a$. (Thus, the definition essentially involves a choice of coproduct.) However, A need not have a cocommutative coproduct in general.

Now suppose that such an algebra acts on a free R-module V (not necessarily of finite rank), and the action is denoted by $a(v)$ as above.

Definition 1.3. Let A^{mult} denote the left A-module A, under left multiplication.

(1) The *smash product* of $TV = T_R V$ and A, denoted by $TV \rtimes A$, is defined to be the A-module $(T_R V) \otimes A$, with the multiplication relations given by

$$a \cdot (v \otimes a') := a(v \otimes a') = \Delta(a)(v \otimes a') = \sum a_{(1)}(v) \otimes a_{(2)} a'$$

where $a, a' \in A,\ v \in V$ - or, in other words,

$$a \cdot v = \sum a_{(1)}(v) a_{(2)}\ \forall a \in A,\ v \in V$$

(2) Suppose A is cocommutative. The smash product of $\operatorname{Sym}_R(V)$ and A, denoted by $\mathscr{H}_0 = (\operatorname{Sym}_R V) \rtimes A$, is defined to be the A-module (explained below) $(\operatorname{Sym}_R V) \otimes A$, with the multiplication given as above.

Remark 1.4. Note that Δ maps 1_A to $1_{A \otimes A}$; this also ensures that $1_A \cdot v = v \cdot 1_A$ in $TV \rtimes A$. Moreover, the above definition naturally suggests the following action of A on TV (under which TV satisfies the axiom of an A-module algebra):

$$a(v_1 \otimes \cdots \otimes v_k) := \sum a_{(1)}(v_1) \otimes \cdots \otimes a_{(k)}(v_k)$$

If we now assume A to be cocommutative in the second definition, and we want to get from $TV \otimes A$ to $(\operatorname{Sym} V) \otimes A$, then the relations that we quotient out by, namely $\wedge^2 V$ (that is a subspace of $V \otimes V$ via: $v \wedge v' \mapsto v \otimes v' - v' \otimes v$), is an A-submodule of $V \otimes V$. This is because we get that $a \cdot (v \wedge v') = \sum a_{(1)}(v) \wedge a_{(2)}(v')$ if $\Delta = \Delta^{op}$.

We now consider a deformation of this latter algebra.

Definition 1.5. (A cocommutative.) Given a skew-symmetric "bilinear form" $\beta \in \operatorname{Hom}_R(V \wedge_R V, A \oplus V)$, the *Drinfeld-Hecke algebra* $\mathscr{H}_\beta$ *(over A) with parameter* β is defined to be the quotient of $TV \rtimes A$ by the relations

$$vv' - v'v =: [v, v'] = \beta(v, v')\ \forall v, v' \in V$$

We define $\beta = \beta_A \oplus \beta_V$, the respective components.

Remark 1.6.

(1) The terminology is mentioned in [Gr, §2]; such algebras were first considered by Drinfeld in [Dr, §4].

(2) More generally, we could consider $\beta_V : V \wedge_R V \to T_R V$. However, we wish to consider algebras satisfying the *PBW property* (explained below); hence the relations should give a filtration on $\mathscr{H}_\beta$.

If we now assign a (common positive) degree to all elements of V, then there is a filtration only if $\operatorname{im} \beta_V \subset T^0 V \oplus T^1 V$. But the degree zero part is already taken care of by β_A (since A contains scalars via the unit). Hence we take $\operatorname{im} \beta_V \subset V$.

Lemma 1.7. *The following relations hold in* $\mathscr{H}_\beta$*:*

(1) (A-compatibility of β in $\mathscr{H}_\beta$.) *The composite*

$$V \wedge_R V \xrightarrow{\beta} A \oplus V \to \mathscr{H}_\beta$$

(denoted also by β here) satisfies:

$$a\beta(v, v') = \sum \beta(a_{(1)}(v), a_{(2)}(v'))a_{(3)}$$

(2) (Jacobi identity in $\mathscr{H}_\beta$.) *Given any $v, v', v'' \in V$, we have*

$$[\beta(v, v'), v''] + [\beta(v', v''), v] + [\beta(v'', v), v'] = 0$$

Proof. That the second relation holds is self-explanatory, since the Jacobi identity holds in any associative algebra under the usual commutator Lie bracket $[a, b] := ab - ba$. The first identity comes from computing

$$a\beta(v, v') = a \cdot (vv' - v'v)$$

in $\mathscr{H}_\beta$ (i.e. via the relations $av = \sum a_{(1)}(v)a_{(2)}$), and using the cocommutativity of A. □

Remark 1.8. We will show that the PBW property is the converse to the above lemma; see Theorem 2.1 below. But first, we write the Jacobi identity carefully:

$$\beta_A(v, \beta_V(v', v'')) + \beta_A(v', \beta_V(v'', v)) + \beta_A(v'', \beta_V(v, v')) = 0 \qquad (1.9)$$

and

$$\begin{aligned} & \beta_V(v, \beta_V(v', v'')) + \beta_V(v', \beta_V(v'', v)) + \beta_V(v'', \beta_V(v, v')) \\ & + v\beta_A(v', v'') + v'\beta_A(v'', v) + v''\beta_A(v, v') \\ = & \sum \beta_A(v, v')_{(1)}(v'')\beta_A(v, v')_{(2)} + \sum \beta_A(v', v'')_{(1)}(v)\beta_A(v', v'')_{(2)} \\ & + \sum \beta_A(v'', v)_{(1)}(v')\beta_A(v'', v)_{(2)} \end{aligned}$$

and these are equations in A and $V \otimes A$ respectively, both mapped into $\mathscr{H}_\beta$. (Note that $V = V \otimes 1 \subset V \otimes A$.)

We will show that the PBW property requires that the Jacobi identity (i.e. the equations (1.9)) must hold in $(V \otimes A) \oplus A$ itself.

1.2. **Example - with a symplectic form.** We now look at an example - here, $V = \mathfrak{h} \oplus \mathfrak{h}^*$ for some free R-module $\mathfrak{h}$ with basis $\{v_i : i \in I\}$, and a dual basis v_i^* for $\mathfrak{h}^*$ (so $\langle v_i^*, v_j \rangle = \delta_{ij}$).

Let us first define a symplectic form ω on V. We let $\mathfrak{h}, \mathfrak{h}^*$ be isotropic subspaces, and $\omega(v_i^*, v_j) := \delta_{i,j}$.

Next, we define our algebra $\mathscr{H}_\beta$ using ω. Fix $\{a_i \in A : i \in I\}$ (that need not be distinct), and similarly, $\{w_i \in V : i \in I\}$. Now define $\beta_{\mathbf{a}} : V \wedge_R V \to A \oplus V$ by

$$\begin{aligned} \beta_{\mathbf{a}}(v, v') &:= 0 \text{ if } v, v' \in \mathfrak{h}, \text{ or } v, v' \in \mathfrak{h}^*; \\ \beta_{\mathbf{a}}(v_i^*, v_j) &:= \delta_{ij}(a_i + w_i) \ \forall i, j \in I \end{aligned}$$

and extend by (bi)linearity and skew-symmetry to all of $V \wedge V$. In other words, for all $i \in I$ and $v \in V$, we have

$$\beta_{\mathbf{a}}(x_i, v) = \omega(x_i, v)(a_i + w_i)$$

where $x_i = v_i$ or v_i^*.

The next result concerns the Jacobi identity in such a setup.

Proposition 1.10. *Say $\beta = \beta_{\mathbf{a}}$ as above. Then the Jacobi identity holds in $(V \otimes A) \oplus A$ if and only if for all $i \neq j$, we have (in $V \otimes A$ and in A):*

$$\begin{aligned} \beta_V(x_j, w_i) \otimes 1 + x_j \otimes a_i &= \sum (a_i)_{(1)}(x_j) \otimes (a_i)_{(2)} \\ \beta_A(x_j, w_i) &= 0 \end{aligned}$$

where $x_j = v_j,\ v_j^*$.

As an easy corollary, we have

Corollary 1.11. *The Jacobi identity holds in $(V \otimes A) \oplus A$ if $\dim_R V \leq 2$.*

(Note that $V \wedge_R V = 0$ whenever $\dim_R V < 2$, so the only nontrivial case is when $\dim_R V = 2$ - but this is trivial from the above result.)

Proof of the proposition. To verify the Jacobi identity, we have to take cyclic sums of iterated commutators of words in $\{v_i, v_i^*\}$. By the definition of $\beta_{\mathbf{a}}$, most of these commutators are zero; thus, the only nontrivial cases when we have to verify the identity, are for (v_j, v_i, v_i^*) (where i may or may not equal j), and similar "dual" collections (i.e. replacing $v_j \leftrightarrow v_j^*$). We first verify the case $j = i$:

$$[[v_i, v_i^*], v_i] + [[v_i^*, v_i], v_i] + [[v_i, v_i], v_i^*] = [-(a_i + w_i), v_i] + [a_i + w_i, v_i] + [0, v_i^*] = 0$$

(since the relations that are used to "evaluate" the first commutator, are exactly the ones used for the second). We next verify the case $j \neq i$, using $x_j = v_j$ or v_j^*:

$$\begin{aligned} & [[v_i, v_i^*], x_j] + [[v_i^*, x_j], v_i] + [[x_j, v_i], v_i^*] \\ = \ & [-(a_i + w_i), x_j] + [0, v_i] + [0, v_i^*] = x_j(a_i + w_i) - (a_i + w_i)x_j \\ = \ & x_j a_i - \sum (a_i)_{(1)}(x_j)(a_i)_{(2)} + \beta_V(x_j, w_i) + \beta_A(x_j, w_i) \end{aligned}$$

Therefore we get that the Jacobi identity is equivalent to the vanishing of these expressions, as was claimed. □

2. Poincaré-Birkhoff-Witt Theorem

2.1. **Statement of the PBW Theorem.** As is standard in such cases, we now consider a different filtered algebra structure on $\mathscr{H}_\beta$: assign degree 1 to V and 0 to A. We say that the *PBW theorem holds for $\mathscr{H}_\beta$* if for any (totally) ordered R-bases $\{x_i : i \in I\}$ of the free R-module V and $\{a \in J_1\}$ of the R-free R-Hopf algebra A, the collection $\{X \cdot a : X$ is a word in the x_i's in nondecreasing order of subscripts, $a \in J_1\}$ is an R-basis of $\mathscr{H}_\beta$.

Equivalently, $\mathscr{H}_\beta$ (or β) has the *PBW property* if the associated graded algebra of $\mathscr{H}_\beta$ (with respect to the $V \mapsto 1, A \mapsto 0$ degree filtration) equals $(\mathrm{Sym}_R V) \rtimes_R A$.

We now state the PBW Theorem. For completeness, we write the full setup here; the last assumption is new.

Theorem 2.1 (PBW Theorem)**.** *Suppose A is an R-free cocommutative R-algebra with coproduct, and V an R-free A-module. Define $TV \rtimes A$ and $\mathscr{H}_\beta$ as above (with $\beta = \beta_A \oplus \beta_V : V \wedge V \to A \oplus V$), and suppose $A = R \cdot 1 \bigoplus A'$ for some free R-submodule A'. Then the following are equivalent:*

1. *β has the PBW property (with $1 \in J_1$).*
2. *The map : $(V \otimes A) \oplus A \to \mathscr{H}_\beta$ is an injection.*
3. *$\beta : V \wedge_R V \to A \oplus V$ satisfies the following two conditions:*
   (a) (A-compatibility of β_A, β_V in $A, V \otimes A$.) *The maps β_A, β_V satisfy:*

$$\begin{aligned} \sum \beta_A(a_{(1)}(v), a_{(2)}(v'))a_{(3)} &= a\beta_A(v, v') \\ \sum \beta_V(a_{(1)}(v), a_{(2)}(v'))a_{(3)} &= \sum a_{(1)}(\beta_V(v, v'))a_{(2)} \end{aligned}$$

   (b) (Jacobi identities in $(V \otimes A) \oplus A$.) *For any $v, v', v'' \in V$,*

$$[\beta(v, v'), v''] + [\beta(v', v''), v] + [\beta(v'', v), v'] = 0$$

   or, more precisely, the two equations (1.9) *hold in A and $V \otimes A$ respectively (identifying V with $V \otimes 1_A \subset V \otimes A$).*

To show the theorem, we will apply the *Diamond Lemma* cf. [Be, Theorem 1.2] - but in a later subsection. For now, we make several remarks.

Remark 2.2.

1. The key difference between Lemma 1.7 and the above theorem is in the ambient spaces in which the relations hold.
2. We need the unit 1 to be one of our basis vectors for A. (This always holds if R is a field.) Words involving this basis vector are to be considered "without" the 1.
3. **Question.** We will say later on, that *Weyl algebras* (i.e. $A = R \cdot 1$) are an example where the PBW property holds. One can also ask for which (possibly Hopf) algebras does the following hold: for all A-modules V, there exists a $\beta : V \wedge V \to R = R \cdot 1 \hookrightarrow A$, with the PBW property.

2.2. **The compatibility condition.** We now note that the A-compatibility implies that β_V is essentially an A-module map in one way, and also a module map for a "particular" unital R-subalgebra of A.

Proposition 2.3. *Fix an R-basis of A containing 1, with respect to which we write $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$ (thus the $a_{(2)}$'s are basis elements, and each $a_{(1)} \in A$). Let A_Δ be the image of $\pi_1 \circ \Delta : A \to A$, where π_1 is the first projection : $A \otimes A \to A$. (Thus, it is the span of all the elements $a_{(1)}$.) Then*

1. *A_Δ is a unital R-subalgebra of A; if A has a counit (i.e. is a coalgebra), $A_\Delta = A$.*
2. *The following are equivalent:*
   - (a) *β_V is A-compatible.*
   - (b) *β_V is an A_Δ-module map.*
   - (c) *$\beta_V \otimes \mathrm{id}_A : V \wedge_R V \otimes A^{mult} \to V \otimes A^{mult}$ is an A-module map.*
3. *β_A is A-compatible if and only if $\widetilde{\beta}_A := mult(\beta_A \otimes \mathrm{id}_A) : V \wedge_R V \otimes A^{mult} \to A^{mult}$ is an A-module map.*

Proof. For the first part, A_Δ is a unital subalgebra because Δ is an algebra map (and $\Delta(1) = 1 \otimes 1$). Next, if (A, Δ) has a counit $\varepsilon : A \to R$, then $a = \sum a_{(1)} \varepsilon(a_{(2)}) \in \sum R a_{(1)} \subset A_\Delta$, for each $a \in A$.

The other two parts are straightforward computations. □

2.3. **Extending the coproduct.** We now examine under what conditions $\mathscr{H}_\beta$ is a (cocommutative) algebra with coproduct - and in particular, when it also has the PBW property.

Definition 2.4. Suppose H is an algebra with coproduct.

1. An element h is *grouplike* if $\Delta(h) = h \otimes h$, and *primitive* if $\Delta(h) = 1 \otimes h + h \otimes 1$. Define $G(H)$ (resp. H_{prim}) to be the set of grouplike (resp. primitive) elements in H.
2. An element $h \in H$ is *skew-primitive* if $\Delta(h) = g \otimes h + h \otimes g'$ for grouplike $g, g' \in G(H)$. Denote the set of such elements by $H_{g,g'}$. (Then $H_{1,1} = H_{prim}$.)

The main result here is

Proposition 2.5. *$\mathscr{H}_\beta$ is a (cocommutative) algebra with coproduct, via the usual operations on A and with V primitive, if $\operatorname{im} \beta_A \subset A_{1,1}$.*

If β has the PBW property, then the converse holds as well.

Proof. Suppose $\operatorname{im} \beta_A \subset A_{1,1}$. Then we extend the map Δ to $\mathscr{H}_\beta$ via: $\Delta(v) = 1 \otimes v + v \otimes 1$ for $v \in V$, and using multiplicativity. One must check that the relations are compatible with Δ now - i.e. that the the ideal I in the "free smash product" $F := TV \# A$, generated by the two additional sets of relations, is a coideal.

First, we verify this for the "a-v" relations:

$$\Delta(av) = \sum (a_{(1)}v \otimes a_{(2)} + a_{(1)} \otimes a_{(2)}v)$$

and

$$\Delta\left(\sum a_{(1)}(v)a_{(2)}\right) = \sum [a_{(1)}(v)a_{(2)} \otimes a_{(3)} + a_{(2)} \otimes a_{(1)}(v)a_{(3)}]$$

and these are equal by the cocommutativity of A, and the "a-v" relations.

For the other computation, one computes that

$$\Delta([v, v']) = [v, v'] \otimes 1 + 1 \otimes [v, v']$$

so if $\beta(v, v')$ is primitive, then these relations are compatible with Δ as well. Since $\beta_V(v, v')$ is always primitive, hence the sufficient condition is that for $\beta_A(v, v')$.

For the converse, we note that the "a-v" relations are satisfied irrespective of β, and we only have to check the "v-v" relations above. By the calculations above (since V is primitive and Δ is multiplicative), we get that

$$\Delta(\beta_A(v, v')) - (1 \otimes \beta_A(v, v') + \beta_A(v, v') \otimes 1) \in (I \otimes (TV\#A)) + ((TV\#A) \otimes I)$$

Now, note that A embeds into $\mathscr{H}_\beta$ by the PBW property, so $A \cap I = 0$ in F. Since the above expression lies in $A \otimes A$ as well, it thus suffices to prove that $(A \otimes A) \cap (I \otimes F + F \otimes I) = 0$ in $F \otimes F$, if $A \cap I = 0$ in F (since this proves that the above expression vanishes, i.e. $\beta_A(v, v')$ is primitive). But this follows by tensoring throughout, to change the base ring to $k(R)$. $\square$

Finally, we once again consider the "undeformed case", $\beta = 0$. Even without the above results (and if A is R-free), the ring $\mathscr{H}_0$ is a cocommutative R-free R-algebra with coproduct (to see this, use that $\Delta_A = \Delta_A^{op}$), with basis given by $\{m \cdot a\}$, where $a \in A$ and m run respectively over some basis of A, and all (monomial) words (including the empty word) with alphabet given by a basis of V.

2.4. **Proof of PBW: preparing to use the Diamond Lemma.** In the rest of this section, we carry out the (somewhat tedious!) computations involved in proving the PBW Theorem. The main tool is the Diamond Lemma, as mentioned above. (Cf. [Gr], it is also possible to mimic the proof of the usual PBW theorem for Lie algebras, but the calculations involved therein are essentially the same.)

Before we use the Diamond Lemma, though, we need to write the algebra relations down systematically - not for $F = TV\#A$, but for the (much larger) free associative R-algebra $T(V \oplus A')$ (where A' was defined in the statement of Theorem 2.1).

Suppose $\{a_j : j \in J\}$ is an R-basis of the R-submodule A' of A, and $J_1 = \{a_j : j \in J\} \coprod \{1_A\}$, with $A = R \cdot 1 \oplus A'$. We write $a_0 = 1$ and $J_0 = J \coprod \{0\}$.

Suppose also that $\{x_i : i \in I\}$ is an R-basis of V. (Here, I is a totally ordered set.) We then define various structure constants, with the right-hand sums (we use Einstein summation convention) running over J_0 and I throughout.

$$\Delta(a_j) = r_j^{kl} a_k \otimes a_l \tag{2.6}$$
$$a_j(x_k) = s_{jk}^i x_i \tag{2.7}$$
$$t_{jk}^{mn} = r_j^{lm} s_{lk}^n \tag{2.8}$$

Then the algebra relations involved here are (once again, the sums in A run over J_0 rather than over J):

$$a_j a_k = u_{jk}^i a_i \tag{2.9}$$
$$x_j x_k = x_k x_j + v_{jk}^i a_i + w_{jk}^h x_h \qquad \forall j > k \tag{2.10}$$
$$a_j x_k = t_{jk}^{mn} x_n a_m \tag{2.11}$$

Thus, the r, s, t, u, v, w's are all structure constants in R, for all choices of indices. Moreover, though the first relation ($a_j a_k = u_{jk}^l a_l$) is only for $j, k \neq 0$, we also define

$$u_{0k}^i = \delta_k^i = u_{k0}^i \tag{2.12}$$

for completeness' sake, since $a_0 a_k = a_k = a_k a_0 \ \forall k$. To see the last equation, we compute, using the above relations:

$$a_j x_k = \sum (a_j)_{(1)}(x_k)(a_j)_{(2)} = r_j^{lm} a_l(x_k) \cdot a_m = r_j^{lm} s_{lk}^n x_n a_m$$

- The first thing to note is that the associative algebra structure of A implies that the u's satisfy an extra condition:

$$a_i(a_j a_k) = a_i \cdot u_{jk}^l a_l = u_{jk}^l u_{il}^m a_m,$$
$$(a_i a_j) a_k = u_{ij}^l a_l \cdot a_k = u_{ij}^l u_{lk}^m a_m$$

so that we get

$$u_{jk}^l u_{il}^m = u_{ij}^l u_{lk}^m \ \forall i, j, k, m \tag{2.13}$$

Let us note down a few other relations. For example, the cocommutativity of A is given by:

$$r_j^{kl} = r_j^{lk} \ \forall j, k, l \in J_0 = J \cup \{0\} \tag{2.14}$$

Next, Δ is multiplicative. Hence

$$\begin{aligned} u_{jk}^l r_l^{mn}(a_m \otimes a_n) &= u_{jk}^l \Delta(a_l) = \Delta(a_j a_k) = \Delta(a_j)\Delta(a_k) \\ &= r_j^{cd} r_k^{ef}(a_c \otimes a_d)(a_e \otimes a_f) = r_j^{cd} r_k^{ef} u_{ce}^m u_{df}^n (a_m \otimes a_n) \end{aligned}$$

Equating coefficients in $A \otimes A$, we conclude that

$$u_{jk}^l r_l^{mn} = r_j^{cd} r_k^{ef} u_{ce}^m u_{df}^n \tag{2.15}$$

Finally, V is an A-module. Hence

$$s_{jm}^n s_{ki}^m x_n = a_j\left(s_{ki}^m x_m\right) = a_j(a_k(x_i)) = (a_j a_k)(x_i) = u_{jk}^m a_m(x_i) = u_{jk}^m s_{mi}^n x_n$$

whereby we get

$$s^n_{jm} s^m_{ki} = u^m_{jk} s^n_{mi} \tag{2.16}$$

- The set of algebra relations above, is what we will denote by S, our *reduction system*. Thus, we need some additional notation: define $X = \{a_j : j \in J\} \cup \{x_i : i \in I\}$. Then expressions in the left- (resp. right-) hand sides in the equations in S, are what Bergman calls f_σ (resp. W_σ).

- Then the expressions in $\langle X \rangle$ (the free semigroup on X) that are *irreducible* are precisely the PBW-basis that was claimed earlier, i.e. words $x_{i_1} \dots x_{i_l} \cdot a_j$, for $j \in J$ and $i_1 \leq i_2 \leq \dots \leq i_l$, all in I. This also includes the trivial word 1.

  Thus, the module $R\langle X \rangle_{\rm irr}$ is precisely the R-span of the above words.

- Next is the notion of a *semigroup partial ordering* on X. Define $\leq$ on the generators of X by:

$$1 < x_i < x_{i'} < a_j \ \forall j \in J,\ i < i' \in I$$

  and extend to a total order on $\langle X \rangle$, by
  - declaring words of length m to be strictly less than words of length n, whenever $m < n$, and
  - ordering words of equal lengths lexicographically.

  This is easily verified to be a semigroup partial order on $\langle X \rangle$. Moreover, this ordering $\leq$ is indeed *compatible with* S, since the relations say that each f_σ is a linear combination of monomials $< f_\sigma$.

- The final item, before the proof of the PBW theorem, is the notion of *ambiguities*. It is clear that no W_σ is a subset of W_τ for some $\sigma, \tau \in S$. Hence there are no *inclusion ambiguities*, and we only have to resolve all *overlap ambiguities*.

2.5. **Finishing the proof.** Before we resolve the overlap ambiguities, we will need the following preliminary result. But first, some notation. The *descending chain condition* means that given a monomial $B \in \langle X \rangle$, no matter what sequence of reductions (i.e. elements of S) we apply to B, we will get to an irreducible expression in $T(V \oplus A')$ in finitely many steps (i.e. beyond some point, we cannot apply any reduction).

Proposition 2.17. *The semigroup partial ordering $\leq$ on $\langle X \rangle$ satisfies the descending chain condition.*

Proof. In fact, we produce an explicit bound for this maximum possible number of reductions applicable on a monomial. So suppose we start with a word $w = T_1 \dots T_n$, where $T_i \in X$ for all i. Define the *misordering index*

$mis(w)$ to be $o + p + pr + q + r^3$, where

$$\begin{aligned} o &= \#\{(i,j) : i < j,\ T_i, T_j \in V,\ T_i > T_j\}, \\ p &= \#\{(i,j) : i < j,\ T_i \in A',\ T_j \in V\}, \\ q &= \#\{i : T_i \in A'\}, \\ r &= \#\{i : T_i \in V\} = n - q \end{aligned}$$

We can now show that each reduction strictly reduces the misordering index of each resulting monomial, which proves the result. □

We now prove the PBW theorem in various steps.

Proof of the PBW Theorem.

(1) The first step is to show the "easier implications" (among the cyclic chain (1) ⇒ (2) ⇒ (3) ⇒ (1)). Clearly, if the PBW property holds, then $(V \otimes A) \oplus A$ injects into $\mathscr{H}_\beta$ - consider the R-span of the monomials $\{x_i \otimes a_j : i \in I, j \in J_0\} \cup \{a_j : j \in J_0\}$.

Next, the compatibility and Jacobi relations hold in $A, V \otimes A$. But since these spaces intersect trivially and inject into $\mathscr{H}_\beta$, we are now done by Lemma 1.7.

(2) To show the last implication - namely, that the compatibility and Jacobi relations imply the PBW property - we assume that compatibility and Jacobi hold in the respective spaces, and prove PBW using the Diamond Lemma. From all the remarks and analysis above, all that we have to do, is to resolve all overlap ambiguities. These are of the following kinds:

$$a_i a_j a_k,\ a_j a_k x_i,\ a_k x_i x_j (i > j),\ x_i x_j x_k (i > j > k)$$

The first kind of overlap ambiguity is resolved because A is an associative algebra (as mentioned above). To complete the proof, one has to systematically analyse the other three types of ambiguities.

We will only analyse the second type of ambiguity above; the others involve carrying out similar (and perhaps more longwinded!) computations, that use various properties of the cocommutative algebra A with coproduct (these properties are written out in terms of the structure constants mentioned above).

We now resolve the ambiguity. From the right-hand side, we have (using the defining relations as well):

$$(a_j a_k) x_i = u_{jk}^m a_m x_i = u_{jk}^m t_{mi}^{lh} x_h a_l = u_{jk}^m r_m^{lc} s_{ci}^h \cdot x_h a_l$$

whereas the left-hand side gives

$$a_j (a_k x_i) = t_{ki}^{fg} a_j x_g a_f = t_{ki}^{fg} t_{jg}^{yh} x_h a_y a_f = t_{ki}^{fg} t_{jg}^{yh} u_{yf}^l x_h a_l$$

Therefore it suffices to show that for all i, j, k, l, h (or h-l), we have

$$u_{jk}^m r_m^{lf} s_{fi}^h = t_{ki}^{fg} t_{jg}^{yh} u_{yf}^l$$

To see this, we start with the right-hand side, expand using the definition of t, and then use equations (2.15), (2.16) above:

$$\begin{aligned} t_{ki}^{fg} t_{jg}^{yh} u_{yf}^{l} &= r_{k}^{fa} s_{ai}^{g} \cdot r_{j}^{yn} s_{ng}^{h} \cdot u_{yf}^{l} = r_{k}^{fa} r_{j}^{yn} u_{yf}^{l} \cdot (s_{ng}^{h} s_{ai}^{g}) \\ &= r_{k}^{fa} r_{j}^{yn} u_{yf}^{l} (u_{na}^{g} s_{gi}^{h}) = r_{j}^{yn} r_{k}^{fa} u_{yf}^{l} u_{na}^{g} \cdot s_{gi}^{h} \\ &= u_{jk}^{m} r_{m}^{lg} \cdot s_{gi}^{h} \end{aligned}$$

and changing the dummy index g to f gives us the expression obtained above from the left-hand side. We have therefore resolved the ambiguity successfully.

□

3. Example: Grouplike (Hecke) algebras

In this section, we analyse an example involving a cocommutative algebra with a coproduct (but no corresponding counit).

3.1. **Generic Hecke algebras.**

Suppose W is a finite Coxeter group with simple reflections S (and length function ℓ), and two R-valued functions a, b defined on conjugacy classes of reflections in W. Then one can define a *generic R-algebra* $\mathcal{E}_W$, as follows (see [Hum, Chapter 7] for more details):

$\mathcal{E}_W$ is generated by $\{T_s : s \in S\}$, with relations for all $s \in S, w \in W$:

$$T_s T_w = \begin{cases} T_{sw}, & \text{if } \ell(sw) > \ell(w); \\ a_s T_w + b_s T_{sw}, & \text{if } \ell(sw) < \ell(w). \end{cases}$$

where $T_w = T_{s_1} \dots T_{s_k}$ is well-defined whenever $w = s_1 \dots s_k$ is a reduced expression in W. Equivalently, $\mathcal{E}_W$ is a free R-module with basis $\{T_w : w \in W\}$, and (valid) associative multiplication, given by

$$\begin{aligned} T_s T_w &= T_{sw} \text{ if } \ell(sw) > \ell(w) \\ T_s^2 &= a_s T_s + b_s T_1 \end{aligned}$$

We want to analyse the case when $A = \mathcal{E}_W$ (so we need $\mathcal{E}_W$ to have a coproduct). It is now easy to see the following

Lemma 3.1. *The map $\Delta : T_s \mapsto T_s \otimes T_s$ extends to make $\mathcal{E}_W$ a (cocommutative) R-algebra with coproduct, if and only if $a_s b_s = a_s(a_s - 1) = b_s(b_s - 1) = 0$ for all s.*

Thus, three obvious solutions are:

(1) $a_s \equiv 0, b_s \equiv 1$, which corresponds to the group algebra (i.e. the Hopf algebra) RW. For more on this case, see e.g. [Dr, EG, Gr].
(2) $a_s \equiv 1, b_s \equiv 0$ - which defines the 0-*Hecke algebra*, cf. [Nor].
(3) $a_s \equiv 0 \equiv b_s$, which defines the *nilCoxeter* (or *nil-Hecke*) *algebra* NC_W.

In the analysis below, we will focus on the last case. However, here is a consequence of A-compatibility in the three cases above.

Proposition 3.2. *In what follows, $s \in S, w \in W, v, v' \in V$, and $\beta : V \wedge_R V \to A = \mathcal{E}_W$.*

1. *If $\mathcal{E}_W = RW$, then the A-compatibility of β is equivalent to: $\forall s, w, v, v'$, $\beta_{sws}(T_s(v), T_s(v')) = \beta_w(v, v')$.*
2. *If $\mathcal{E}_W = NC_W$, and β is A-compatible, then β_w is identically 0 if $\ell(w) < \ell(sw) < \ell(sws)$ in W, for some $s \in S$.*
3. *Suppose $\mathcal{E}_W$ is the 0-Hecke algebra, β is A-compatible, and $\ell(w) < \ell(sw) < \ell(sws)$ for some $w, s \in W$. Then $\beta_w \equiv -\beta_{sw}$ on $V \wedge V$ and $\beta_{sws} \equiv -\beta_{sw}$ on $\operatorname{im}(T_s) \wedge \operatorname{im}(T_s)$.*

Proof. We consider the three cases separately.

1. This is easy to prove.
2. For notational convenience we use $>$ to also denote the Bruhat order: $sw > w$ if $\ell(sw) > \ell(w)$, and similarly for $ws > w$ (where $s \in S, w \in W$).

   We write out the condition, with $a = T_s$ for $s \in S$:
$$T_s \beta(x, y) = \beta(T_s(x), T_s(y)) T_s$$
   or, expanding using the multiplication in NC_W,
$$\sum_{w : sw > w} \beta_w(x, y) T_{sw} = \sum_{w \in W} \beta_w(T_s(x), T_s(y)) T_w T_s$$
   Now relabel the right-hand sum using sws in place of w. Thus, it becomes
$$\sum_{w \in W} \beta_{sws}(T_s(x), T_s(y)) T_{sws} T_s = \sum_{w : sw > sws} \beta_{sws}(T_s(x), T_s(y)) T_{sw}$$
   Equating coefficients, we get that if $w < sw < sws$, then $\beta_w(x, y) T_{sw} = 0$ for all x, y.
3. In the case of the 0-Hecke algebra, note that left-multiplication by a fixed $s \in S$ partitions W into two disjoint subsets, namely
$$L_s := \{w \in W : \ell(sw) > \ell(w)\} \leftrightarrow W \setminus L_s = \{w \in W : \ell(sw) < \ell(w)\}$$
   and similarly for right-multiplication ($R_s \coprod (W \setminus R_s) = W$). Thus, the A-compatibility implies
$$T_s \beta(v, v') = \beta(T_s(v), T_s(v')) T_s$$
   We now look on the left side. We evidently have that it equals
$$\sum_{w \in L_s} (\beta_w(v, v') + \beta_{sw}(v, v')) T_{sw}$$
   by definition of L_s and the defining relations in the 0-Hecke algebra. Similarly, the right-hand side equals
$$\sum_{w \in R_s} (\beta_w(v, v') + \beta_{ws}(v, v')) T_{ws}$$

Now, if $w < sw < sws$, then $L_s \ni w \leftrightarrow sw$ and $R_s \ni sw \leftrightarrow sws$. This means that T_{sw} has no coefficient on the right-side above, and T_{sws} has no coefficient on the left-hand side above. Hence

$$\beta_w(v,v') + \beta_{sw}(v,v') \equiv 0 \equiv \beta_{sw}(T_s(v), T_s(v')) + \beta_{sws}(T_s(v), T_s(v'))$$

as claimed.

□

3.2. **Grouplike algebras and the Jacobi identity.** We now unite the above cases by introducing the following notion.

Definition 3.3. Given a group G, a *grouplike algebra (of type G)* is an R-algebra $\mathcal{E}_G$ together with a set of elements $\{T_g : g \in G\}$, so that:

(1) $\{T_g : g \in G\}$ is a basis of the free R-module $\mathcal{E}_G$.
(2) T_1 is the unit in $\mathcal{E}_G$.
(3) The map $\Delta : \mathcal{E}_G \to \mathcal{E}_G \otimes \mathcal{E}_G$, given by $T_g \mapsto T_g \otimes T_g$, makes $\mathcal{E}_G$ an algebra with coproduct.

(In other words, these are cocommutative algebras with coproduct, that have a free R-module basis $\{T_g : g \in G\}$ of grouplike elements, with T_1 the multiplicative identity as well.)

Thus, group rings and the generic Hecke algebras in Lemma 3.1 above, are examples of (cocommutative) grouplike algebras (with coproduct).

Standing Assumption 1. For the rest of this section, R is a field and $\beta = \beta_A$.

Setup: Now suppose V is an $\mathcal{E}_G$-module, and we once again define $\mathscr{H}_\beta$ as above. Write $\beta(x,y) = \beta_A(x,y) = \sum_{g \in G} \beta_g(x,y) T_g$ for $x, y \in V$; thus, β_g is a skew-symmetric bilinear form on V, for all $g \in G$.

We also define

$$d_g := \dim_R \operatorname{im}(T_1 - T_g) = \operatorname{codim}_V \ker(T_1 - T_g) = \operatorname{codim}_V \operatorname{Fix}(T_g)$$

where $\operatorname{Fix}(T_g)$ is the *fixed point space* $\{v \in V : T_g(v) = v\}$. Finally, $\operatorname{Rad}(\beta_g)$ is defined to be the *radical* of the bilinear form: $\operatorname{Rad}(\beta_g) := \{v \in V : \beta_g(v, V) \equiv 0\}$.

We now characterize the Jacobi identity in this general setting.

Theorem 3.4. *The Jacobi identity holds in $\mathscr{H}_\beta$ if and only if for all $g \in G$, one of the following three conditions holds:*

(1) $\beta_g \equiv 0$.
(2) $T_g \equiv \mathrm{id}|_V$, *i.e.* $d_g = 0$.
(3) d_g *is 1 or 2, and* $\operatorname{Rad}(\beta_g)$ *is a subspace of* $\operatorname{Fix}(T_g)$, *of codimension* $2 - d_g$.

Proof. We explicitly write out one term of the Jacobi identity:

$$\begin{aligned}[\beta(x,y),z] &= \sum_{g\in G}\beta_g(x,y)(T_gz - zT_g) = \sum_{g\in G}\beta_g(y,x)(z - T_g(z))T_g \\ &= \sum_{g\in G}\beta_g(y,x)(T_1 - T_g)(z)T_g\end{aligned}$$

Now write out the cyclic sum and equate the coefficients of $\otimes T_g$ for each g:

$$\beta_g(y,x)(T_1 - T_g)(z) = \beta_g(y,z)(T_1 - T_g)(x) + \beta_g(z,x)(T_1 - T_g)(y) \quad (3.5)$$

It is this equation that must be satisfied, for all g. We now prove both implications.

The "only if" part. Suppose the Jacobi identity holds. We assume that β_g is not identically zero. Now choose x, y so that $\beta_g(y,x) \neq 0$. Considering the above equation for all z (note that R is a field),

$$\operatorname{im}(T_1 - T_g) \subset Rx' + Ry'$$

where $x' = (T_1 - T_g)(x)$ and $y' = (T_1 - T_g)(y)$. In particular, $\dim_R \operatorname{im}(T_1 - T_g) \leq 2$ if $\beta_g \neq 0$. There are three cases now: $d_g = 0, 1, 2$. But before we analyse them, we note that if we plug in $z \in \operatorname{Fix}(T_g)$ in equation (3.5), then we get that

$$\beta_g(y,z)x' + \beta_g(z,x)y' = 0 \quad (3.6)$$

(1) If $d_g = 0$, there is nothing to prove, since $T_g \equiv T_1 = \operatorname{id}|_V$, and equation (3.5) trivially holds.

(2) If $d_g = 2$, then x', y' are necessarily linearly independent. We first show that $\operatorname{Fix}(T_g) \subset \operatorname{Rad}(\beta_g)$: given $z \in \operatorname{Fix}(T_g)$ in (3.6), we get that $\beta_g(z,y) = \beta_g(z,x) = 0$. Moreover, if we plug in z, z', x in equation (3.5) above, with $z, z' \in \operatorname{Fix}(T_g)$ and x as above, then we get that $\beta_g(z,z')(T_1 - T_g)(x) = 0$. Thus, $\beta_g(z,z') = 0$ for all $z' \in \operatorname{Fix}(T_g)$, whence $z \in \operatorname{Rad}(\beta_g)$, as desired.

Conversely, if we choose $z \in \operatorname{Rad}(\beta_g)$ and x, y as above, then equation (3.5) gives that $\beta_g(y,x)(T_1 - T_g)(z) = 0$, whence $z \in \operatorname{Fix}(T_g)$. Thus $\operatorname{Fix}(T_g) = \operatorname{Rad}(\beta_g)$ as desired.

(3) The final case is when $d_g = 1$. Take any $v_1 \notin \operatorname{Fix}(T_g)$; if $v_1 \in \operatorname{Rad}(\beta_g)$, then since we assumed that β_g is not identically zero, hence $\beta_g(v_0, v_0') \neq 0$ for some $v_0, v_0' \in \operatorname{Fix}(T_g)$. But then $\beta_g(v_1 + v_0, v_0') \neq 0$.

Similarly, if $\beta_g(v_1 + v_0, v_1 + v_0') \neq 0$ for v_1, v_0, v_0' as above, then $\beta_g(v_1 + v_0, v_0' - v_0) \neq 0$. Therefore we can assume without loss of generality, that $\operatorname{Fix}(T_g) = Rv_0 \oplus V^\perp$ and $V = Rv_1 \oplus \operatorname{Fix}(T_g)$, with $\beta_g(v_1, v_0) \neq 0$ and $\beta_g(v_1, V^\perp) = 0$.

The result is proved, if we show that $V^\perp = \operatorname{Rad}(\beta_g)$. So take $z \in V^\perp, y \in \operatorname{Fix}(T_g)$, and $x = v_1$ in equation (3.5); thus $\beta_g(y,z) = 0$. By choice of $V^\perp$, we have $\beta_g(z,x) = 0$. Finally, $\beta_g(v_1, v_0) \neq 0$, and we are done.

As a side-consequence, we have shown that $\mathrm{Fix}(T_g)$ is actually isotropic if the Jacobi identity holds and $d_g = 1, 2$ or $\beta_g \equiv 0$. Moreover, if the given (and supposedly equivalent) conditions hold, then the third condition suggests that $\mathrm{Rad}(\beta_g)$ has codimension at most one in $\mathrm{Fix}(T_g)$, whence $\mathrm{Fix}(T_g)$ is again isotropic with respect to β_g. We will use this below.

The "if" part. Conversely, if $\beta_g \equiv 0$ or $T_g \equiv T_1 = \mathrm{id}$ for any $g \in G$, then equation (3.5) holds. It remains to verify that it also holds in the two remaining cases: $d_g = 1, 2$. The proof is in two parts.

The first case is when no two of x, y, z are linearly independent modulo $\mathrm{Fix}(T_g)$ (in either of the two cases $d_g = 1, 2$). We then write $\mathrm{Fix}(T_g) = Rv_0 \oplus V^\perp$, and choose $x \notin \mathrm{Fix}(T_g)$, so that $\mathrm{Fix}(T_g)$ is β_g-isotropic, and $V^\perp$ or $\mathrm{Fix}(T_g)$ equals $\mathrm{Rad}(\beta_g)$ - depending on whether d_g is 1 or 2 respectively.
In this case, we write $y = dx + v$ and $z = d'x + v'$, where $d, d' \in R$ and $v, v' \in \mathrm{Fix}(T_g)$. Equation (3.5) now reduces to having to verify that

$$\beta_g(x,v)(T_1-T_g)(d'x)+\beta_g(dx+v,d'x+v')(T_1-T_g)(x)+\beta_g(v',x)(T_1-T_g)(dx)$$

vanishes. After suitable cancellations, we are left to verify that $\beta_g(v, v')(T_1 - T_g)(x) = 0$. But this follows because in both these cases, we showed (above) that $\mathrm{Fix}(T_g)$ is β_g-isotropic.

The second part is similar to the proof of [Gr, Corollary 2.4]. If x, y are linearly independent modulo $\mathrm{Fix}(T_g)$, then $d_g = 2$ and $\mathrm{Rad}(\beta_g) = \mathrm{Fix}(T_g)$, so $\beta_g(x, y) \neq 0$. Then for any $z \in V$, we have

$$z = \frac{\beta_g(z,y)}{\beta_g(x,y)}x + \frac{\beta_g(z,x)}{\beta_g(y,x)}y \mod \mathrm{Fix}(T_g)$$

Multiplying throughout by $\beta_g(x, y)$ and applying $T_1 - T_g$ yields equation (3.5) in this case too. ☐

We end this subsection with a corollary that we will use later.

Corollary 3.7. *Suppose some T_g is nilpotent, and the Jacobi identity holds. Then $\beta_g \equiv 0$ or $\dim_R V = 2$.*

Proof. From the above theorem, since T_g is nilpotent, hence $T_1 - T_g = \mathrm{id} - T_g$ is invertible, whence $d_g = \dim_R V$. So either $\beta_g \equiv 0$, or $d_g = \dim_R V$ is 1 or 2. But if $\dim_R V = 1$, then $\beta : V \wedge_R V \to A$ is zero. ☐

3.3. **PBW property for nilCoxeter algebras.** We first remark that the PBW property is characterized (in the case $A = RW$ for a finite group W) in [Gr, Theorem 2.3, Corollary 2.4], more or less in terms of Theorem 3.4, and the first part of Proposition 3.2 above.

For the rest of this section, fix $G = W$, a finite Coxeter group, and $A = \mathcal{E}_G = \mathcal{E}_W = NC_W$, the nilCoxeter algebra (as in the subsection on generic Hecke algebras above). We now define $\mathscr{H}_\beta$ as usual (given V), and characterize the β's that have the PBW property.

Next, note that NC_W does not have a counit ε; otherwise we would have $T_s\varepsilon(T_s) = T_s$, whence $\varepsilon(T_s) = 1$ - but $\varepsilon(T_s)^2 = \varepsilon(T_s)^2 = 0$. Thus, this gives us an algebra with coproduct $\Delta = \Delta^{op}$, without a counit.

Our aim is to try and understand what the PBW Theorem says in this case.

Theorem 3.8. *Setup as above.*

1. *The Jacobi identity holds in $\mathscr{H}_\beta$ if and only if $\dim V \leq 2$, or $\operatorname{im}\beta \subset R = R \cdot T_1$.*
2. *$\mathscr{H}_\beta$ has the PBW property if and only if $\dim V = 2$ and $\operatorname{im}\beta \subset R \cdot T_{w_\circ}$, or $\beta = 0$. Here, $w_\circ$ denotes the (unique) longest element in W.*

To prove this, we need some results on modules over NC_W. First, note that NC_W has an augmentation ideal $N^+ := \sum_{w \neq 1} RT_w$. Moreover, there is only one simple module upto isomorphism: $R \cong NC_W/N^+$. This is because we will show that every module (and thus every simple module) contains a one-dimensional submodule that is killed by N^+.

We now define a nonzero vector $v \neq 0$ in an NC_W-module to be *primitive* if $N^+v = 0$. Let $\operatorname{Prim}(V)$ denote the set of primitive vectors in V, and the zero vector.

Proposition 3.9. *Suppose V is an NC_W-module.*

1. *Then $\operatorname{im}(T_w) = V$ if and only if $w = 1$; otherwise T_w is nilpotent.*
2. *Every nonzero V has a primitive vector. Moreover, $\operatorname{Prim}(V)$ is a direct sum of one-dimensional simple modules (and contains $T_{w_\circ}V$).*

Proof.

1. Given $w \neq 1$, find $n > 0$ such that $\ell(w^n) = n\ell(w)$ but $\ell(w^{n+1}) < (n+1)\ell(w)$ (such an n exists, since W is finite). Then $T_{w^n} = (T_w)^n \neq 0 = (T_w)^{n+1}$ (in particular, T_w cannot be a linear isomorphism).
2. Given a nonzero element $v \in V$, let w be any element of largest possible length, so that $T_w v \neq 0$. Then $T_w v$ is primitive. The remaining assertions are obvious.

□

We are now ready to complete the proof.

Proof of Theorem 3.8.

1. If $\dim_R V \leq 2$ then the Jacobi identity holds by Corollary 1.11 above. If $\operatorname{im}\beta \subset R \cdot T_1$ (for general V), then equation (3.5) above shows that the Jacobi identity holds, since $\beta_w \equiv 0$ unless $w = 1$, when $T_1 - T_w = 0$.

   Conversely, suppose the Jacobi identity holds. Then by Theorem 3.4, $\beta_w \equiv 0$ whenever $\dim_R \operatorname{im}(T_1 - T_w) > 2$. But by Proposition 3.9, T_w is nilpotent if $w \neq 1$, so by Corollary 3.7, either $\dim_R V \leq 2$, or $\beta_w \equiv 0$ whenever $w \neq 1$. But this (latter) means that $\operatorname{im}\beta \subset R \cdot T_1$.

(2) We use Theorem 2.1 and the previous part. Let us finish one part of the analysis: suppose that $\dim_R V = 2$. By Proposition 3.9 we write $V = Rx \oplus Ry$, with y primitive, and $\beta(x,y) = \sum_{w \in W} r_w T_w$ for scalars r_w.

Suppose that $r_w \neq 0$ for some w. If we can now choose s so that $sw > w$, then $T_s \beta(x,y) = r_w T_{sw} + \cdots \neq 0$. But the A-compatibility condition suggests that

$$T_s \beta(x,y) = \beta(T_s(x), T_s(y)) T_s = 0$$

since y is primitive. This contradicts the existence of such an s, which implies that the only choice is $w = w_\circ$. Hence $\operatorname{im} \beta \subset R \cdot T_{w_\circ}$. Conversely, this does imply that β is A-compatible, since both sides in the compatibility condition

$$T_w \beta(x,y) = \beta(T_w(x), T_w(y)) T_w$$

vanish if $w \neq \mathrm{id}$, and coincide if $w = \mathrm{id}$.

Next, if $\beta = 0$ (whether or not $\dim_R V < 2$), then PBW holds for our algebra $\mathscr{H}_\beta$. By the previous part, it remains to show that if $\operatorname{im} \beta \subset R \cdot T_1$ (for any V), then β is A-compatible only if $\beta = 0$. But if we choose any $x, y \in V$ and $s \in S$, then

$$T_s \beta(x,y) = \beta(T_s(x), T_s(y)) T_s$$

which gives that $\beta(x,y) = \beta(T_s(x), T_s(y))$ (comparing coefficients) for all x, y. Now replace x, y by $x' = T_s(x), y' = T_s(y)$ respectively; then $T_s^2 = 0$ gives us that

$$\beta(x,y) = \beta(T_s(x), T_s(y)) = \beta(0(x), 0(y)) = 0 \ \forall x, y \in V$$

as desired.

□

4. The case of Hopf algebras

We now specialize our setup to the case that occurs widely in the literature. We revert to our original definition of β.

Standing Assumption 2. Henceforth, A is an R-free cocommutative R-Hopf algebra, V is an A-module, and $\beta : V \wedge_R V \to A \oplus V$.

Thus, A is an A-module under the adjoint action; $TV \otimes A$ and $TV \rtimes A$ are both (A-modules, as well as) A-(Hopf-)module algebras. The (common) action is mentioned in Proposition 4.1 below.

Moreover, in the first part of Lemma 1.7 above, the second arrow takes $a \oplus v \mapsto \overline{a} + \overline{v} \in \mathscr{H}_\beta$, and $A, V, \mathscr{H}_\beta$ are A-modules ($A, \mathscr{H}_\beta$ under the adjoint action). Then the second arrow is an A-module map because of the relations $\operatorname{ad} a(v) = a(v)$ in $\mathscr{H}_\beta$.

4.1. **Relations.** Note that one of the defining relations for $\mathscr{H}_\beta$ can be rephrased, as the following result shows (the proofs are straightforward).

Proposition 4.1. *Suppose some R-Hopf algebra A acts on a free R-module V, and an R-algebra B contains (the images of generators) A, V (possibly among others).*

(1) *Then the following relations are equivalent (in B) for all $v \in V$:*
  (a) $\sum a_{(1)} v S(a_{(2)}) = a(v)$ *for all $a \in A$.*
  (b) $av = \sum a_{(1)}(v) a_{(2)}$ *for all $a \in A$.*

  If A is cocommutative, then both of these are also equivalent to:
  (c) $va = \sum a_{(1)} S(a_{(2)})(v)$ *for all $a \in A$.*

  Now say that this holds.

(2) *Suppose A is cocommutative. Then $\tau : A \otimes V \to V \otimes A$, given by $a \otimes v \mapsto \sum a_{(1)}(v) \otimes a_{(2)}$, as well as $\tau^{op} : V \otimes A \to A \otimes V$, given by $v \otimes a \mapsto \sum a_{(1)} \otimes S(a_{(2)})(v)$, are A-module isomorphisms that are inverse to one another.*

(3) *Any unital subalgebra M of B that is also an A-submodule (via* ad*), is an A-(Hopf-)module algebra under the action*

$$a(m) := \operatorname{ad} a(m) = \sum a_{(1)} m S(a_{(2)}) \ \forall a \in A, \ m \in M$$

For example, in the last part, we can take $M = B$ or A - e.g. $M = TV \rtimes A$ or $\mathscr{H}_\beta$ above.

Later on, we will use the following result, that can (essentially) be found in [Jo, Lemma 1.3.3].

Lemma 4.2. *Given an R-algebra map $\varphi : A \to B$, where A is a Hopf algebra, the centralizer of $\varphi(A)$ in B is the weight space B_ε (where B is an A-module via: $a \cdot b := \sum \varphi(a_{(1)}) b \varphi(S(a_{(2)}))$).*

Consequently, $\mathscr{H}_\beta$ is commutative if and only if $A = A_\varepsilon, V = V_\varepsilon$ (under the adjoint and given actions respectively), and $\beta \equiv 0$.

We conclude this subsection with the following results on anti-involutions. The proofs are straightforward.

Proposition 4.3.

(1) *S extends to an anti-involution of $\mathscr{H}_\beta$ that fixes V, if $\beta_V \equiv 0$ and $S \circ \beta_A = -\beta_A$ (for instance, if $\beta_A(v, v')$ is always primitive in A).*
(2) *$S|_A \cup (-\operatorname{id})|_V$ extends to an anti-involution on $\mathscr{H}_\beta$, if and only if $S \circ \beta_A = -\beta_A$.*
(3) *Suppose $\beta = \beta_{\mathbf{a}}$ as in Proposition 1.10 above. Then S extends to an anti-involution of $\mathscr{H}_\beta$ that interchanges v_i and v_i^* for all i, if and only if (in $\mathscr{H}_\beta$) $S : V \to V$ is an A-module map, and each a_i and w_i is fixed by S.*

4.2. **Yetter-Drinfeld condition.** One of our conditions for the PBW property to hold is equivalent to a compatibility condition called the *Yetter-Drinfeld condition* (e.g. cf. [BaBe, Theorem 3.3]).This is shown in the following result, which generalizes Proposition 2.3 above. In the result below, $\tau^{op} : M \otimes A \to A \otimes M$ is defined just as it was in Proposition 4.1 above, and A^{ad}, A^{mult} refer to different A-module structures on A (via the adjoint action, and via left multiplication respectively).

Proposition 4.4. *Suppose we have $\beta : V \wedge_R V \to M$ (an A-module), and B is an (associative) R-algebra (under mult) containing A, M, with the additional relations $m \cdot a = mult(\tau^{op}(m \otimes a))$ in B. Then the following are equivalent (in B):*

(1) *$\beta : V \wedge_R V \to M$ is* A-equivariant, *or an A-module map:*

$$a(\beta(v, v')) = \sum \beta(a_{(1)}(v), a_{(2)}(v')) \ \forall a \in A, v, v' \in V$$

(2) *β satisfies the* Yetter-Drinfeld (compatibility) condition, *i.e.*

$$\tau^{op}\left(\sum \beta(a_{(1)}(v), v')a_{(2)}\right) = \sum a_{(1)}\beta(v, S(a_{(2)})(v')) \ \forall a \in A, v, v' \in V$$

(3) *β is* A-compatible*: $a\beta(v, v') = \sum \beta(a_{(1)}(v), a_{(2)}(v'))a_{(3)} \ \forall a, v, v'$.*

(4) *$\beta \otimes \mathrm{id}_A : (V \wedge_R V) \otimes A^{mult} \to M \otimes A^{mult}$ is an A-module map.*

(5) *β satisfies: $\beta(v, v')a = \sum a_{(1)}\beta(S(a_{(2)})(v), S(a_{(3)})(v')) \ \forall a, v, v'$.*

If β also satisfies: $\beta(a(v), v') = \beta(v, S(a)(v'))$ for all v, v', a, then these are also equivalent to:

(6) *$\mathrm{im}\, \beta \subset \mathfrak{Z}_B(A)$, i.e. $\beta(v, v')$ commutes (in B) with all of A, $\forall v, v'$.*

The proof is straightforward; it uses Proposition 4.1, Lemma 4.2 and that A is cocommutative.

Remark 4.5.

(1) The result applies to $\beta = \beta_A, \beta_V$. For instance, if $M = A^{ad}$ and $\beta_V \equiv 0$, then we can choose B to be A, in which case the other assumptions are trivially satisfied for the (module structure given by the) adjoint action.

(2) We may ask where the Yetter-Drinfeld condition comes from - or can be seen in. The answer is as follows: in the associative algebra B above, compute $v' \cdot a \cdot v$ in two different ways (i.e. using the maps τ, τ^{op}, β). Then we get that

$$\begin{aligned} & \sum a_{(1)}\beta(v, S(a_{(2)})(v')) = v'av - \sum a_{(1)}(v)a_{(2)}S(a_{(3)})(v') \\ = \ & \sum \beta(a_{(1)}(v), v')a_{(2)} = \tau^{op}\left(\sum \beta(a_{(1)}(v), v')a_{(2)}\right) \end{aligned}$$

which is the Yetter-Drinfeld condition (see the proof of [BaBe, Theorem 3.3]).

4.3. **Hopf algebra structure.** We now explore when $\mathscr{H}_\beta$ has a Hopf algebra structure. We need some more notation.

Definition 4.6. Suppose H is an R-Hopf algebra. A "subspace" $J \subset H$ (J, H are both R-free here) is *weight-stable* if $S(J) \subset J$, $\varepsilon(J) = 0$, and whenever $\mu, \mu' \in \Gamma_H$ kill J, so does $\mu * \mu'$.

(Here, $*$ denotes convolution in $\Gamma_H \subset H^*$.) Also recall the notion of skew-primitive elements in H (see Definition 2.4). One now has the following easy-to-show

Lemma 4.7. *Suppose H is a Hopf algebra.*

1. *If $h \in H_{g,g'}$, then $\varepsilon(h) = 0$, $S(h) = -g^{-1}h(g')^{-1} \in H_{(g')^{-1},g^{-1}}$.*
2. *If J_i is weight-stable in H for all $i \in I$, so is $\sum_{i \in I} J_i$.*

as well as the following *examples* of weight-stable subspaces in H:

1. $J = 0$.
2. Any "subspace" (R-submodule) of $H_{1,1}$.
3. $R(g - g^{-1})$ for any $g \in G(H)$.
4. Any S-stable R-submodule $P_{g,g'}$ of $H_{g,g'} + H_{(g')^{-1},g^{-1}}$. (Note that the previous example is one such: $g - g^{-1} \in H_{g,g^{-1}} \cap H_{g^{-1},g}$.)
5. More generally, using the above lemma, $\sum_{(g,g') \in U} P_{g,g'}$ is weight-stable, for any $U \subset G(H) \times G(H)$ and any choice of S-stable $P_{g,g'}$'s.

Proposition 4.8. *Setup as above.*

1. *The set of weights of $\mathscr{H}_\beta$ is*
$$\Gamma := \{\gamma \in V^* \times \Gamma_A : \gamma \circ \beta = 0,\ \gamma|_V \in V_\varepsilon^*\} = (V_\varepsilon^* \times \Gamma_A) \cap (\operatorname{im}\beta)^\perp$$
and this is a subset of the abelian group $V^* \times \Gamma_A$.
2. *If $\beta_V \equiv 0$, then Γ is a (abelian) subgroup (under $* = (+_{V^*}, *_A)$) if and only if $\operatorname{im}\beta$ is weight-stable in A.*
3. *$\mathscr{H}_\beta$ is a (cocommutative) Hopf-algebra via the usual operations on A (and making V primitive) if $\operatorname{im}\beta_A \subset A_{1,1}$.*

We remark that by the above examples, $\operatorname{im}\beta_A$, being a subset of $A_{1,1}$, is automatically weight-stable, which is compatible with the previous part. Moreover, $\mathscr{H}_0$ is now an R-Hopf algebra, with the set (group) of weights $V_\varepsilon^* \times \Gamma_A$.

Proof. The first two parts are straightforward, and the last part is similar to the proof of Proposition 2.5 above. □

We conclude with a result about when $\mathscr{H}_\beta$ has the PBW property *and* is also a Hopf algebra. This also shows that the "sufficient condition" for general $\mathscr{H}_\beta$, that was mentioned in Proposition 4.8 above, is indeed the "best possible" (when PBW holds).

Theorem 4.9. *Suppose $\mathscr{H}_\beta$ has the PBW property. Then it is a (cocommutative) Hopf algebra (via the usual operations on A, and with V primitive) if and only if $\beta_A(v, v')$ is primitive for all $v, v' \in V$ - i.e.* $\operatorname{im} \beta_A \subset A_{1,1}$.

Proof. One part follows from Proposition 4.8, and the other from Proposition 2.5 above. □

5. Examples

We now mention several examples. In what follows, whenever we say that $\mathscr{H}_\beta$ is (or is not) a Hopf algebra, we implicitly assume that V should be primitive in it.

5.1. **Hopf algebras.** The first example is the "most degenerate" one: take $V = 0$. Then $\mathscr{H}_\beta = \mathscr{H}_0 = A$.

5.2. **Lie algebras.** A different way (than the previous example) to get $\mathscr{H}_\beta = \mathfrak{U}\mathfrak{g}$ for a Lie algebra $\mathfrak{g}$, is to take $A = \mathfrak{U}\mathfrak{h}$ for some Lie subalgebra $\mathfrak{h}$ (e.g. $\mathfrak{h} = 0$) and V to be any vector space complement to $\mathfrak{h}$ in $\mathfrak{g}$. Then β_V, β_A give the Lie bracket on $[V, V]$. Note that $\operatorname{im} \beta_A \subset A_{1,1} = \mathfrak{h}$, and $\mathscr{H}_\beta = \mathfrak{U}\mathfrak{g}$ is indeed a Hopf algebra. If $\mathfrak{h} = 0$ and $V = \mathfrak{g}$, then $\beta = \beta_V$ is the Lie bracket, and the PBW Theorem is equivalent to the Jacobi identity.

5.3. **Weyl algebras.** Let $A = R \cdot 1$ (so that the comultiplication is: $\Delta(1) = 1 \otimes 1$). Then Proposition 1.10 holds with $a_i = 1, w_i = 0\ \forall i$, and so does the PBW property. Moreover, it is not a Hopf algebra with V primitive.

5.4. **The two-dimensional case.** Suppose $V = Rx \oplus Ry$. Then $V \wedge_R V$ is a free rank-one module $R(y \wedge x)$, hence so is $\operatorname{im} \beta$ (the relation here is $[y, x] = \gamma x + \delta y + a$, where $\gamma, \delta \in R, a \in A$). Thus, $y \wedge x$ and $\operatorname{im} \beta$ are "weight vectors" for A, and of the same weight if β is an A-module map.

By Proposition 1.10, the Jacobi identity always holds in $(V \otimes A) \oplus A$. Thus, to check if $\mathscr{H}_\beta$ has the PBW property or not, it is enough to verify whether or not $\beta_A : V \wedge_R V \to A$ is an A-module map.

5.5. **Deformation of Kleinian singularities.** This is from [CBH] and [Kh2, §5.4]: Let $V = Rx \oplus Ry$ and A be the group ring RW of a subgroup W of $GL_2(R)$ (this makes V a W-module). One observes that W preserves the symplectic form $\omega(y, x) = 1$ on V (or, $\omega(bx + dy, ax + cy) \equiv ad - bc$) if and only if $W \subset SL_2(R)$. (Both constructions in the above references are special cases of this.)

Lemma 5.1. $[g(y), g(x)] = \det(g)[y, x]$ *for all* $g \in GL_2(R)$.

Proof. If we write $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $g(x) = g(1,0)^T = ax + cy$, $g(y) = g(0,1)^T = bx + dy$, so that

$$[g(y), g(x)] = [bx + dy, ax + cy] = (da - bc)[y, x] = \det(g)[y, x]$$

for all $g \in GL_2(R)$. □

We now define $\mathscr{H}_\beta = \mathscr{H}_{\gamma,\delta,\lambda}$ via the relation $[y,x] = \gamma x + \delta y + \lambda$, where $\lambda := \sum_{g \in W} a_g g$ (a finite sum), and $\gamma, \delta \in R$. By Proposition 1.10, the Jacobi identity always holds. The following results are now straightforward.

Proposition 5.2.

(1) *$\mathscr{H}_\beta$ has the PBW property if and only if*
   (a) *For all w, $w\lambda w^{-1} = \det(w)\lambda$ in RW, i.e. λ is in the weight space $(RW)_{\det}$.*
   (b) *If $\gamma \neq 0$, then the entries of the 2×2 matrices $w \in W$ satisfy: $w_{22} = 1, w_{21} = 0$ for all $w \in W$.*
   (c) *If $\delta \neq 0$, then the entries of $w \in W$ satisfy: $w_{11} = 1, w_{12} = 0$, for all $w \in W$.*

(2) *Given the PBW property, $\mathscr{H}_{\gamma,\delta,\lambda}$ is a Hopf algebra if and only if $\lambda = 0$.*

Corollary 5.3. *Suppose $W \subset SL_2(R)$. Then λ is central if PBW holds. Conversely, if $\gamma = \delta = 0$ and $\lambda \in \mathfrak{Z}(RW)$, then $\mathscr{H}_\beta$ has the PBW property.*

Note that the constuctions in [CBH] and [Kh2] do indeed have $W \subset SL_2(R)$, $\beta_V \equiv 0$, and $\lambda \in \mathfrak{Z}(RW)$. (Hence PBW holds too.)

5.6. **Symplectic reflection algebras.** These were introduced in [EG]. We let V be finite-dimensional over $\mathbb{C}$ and $A = \mathbb{C}\Gamma$, the group algebra of a finite group $\Gamma \subset GL(V)$. Then $\beta = \kappa : V \wedge_{\mathbb{C}} V \to \mathbb{C}\Gamma$ (so $\beta_V = 0$). The κ's satisfying the PBW property were classified in [EG].

5.7. **Rational Cherednik algebras.** We just mention that this is a special case of the above example of symplectic reflection algebras, that is generated by a finite group $W \subset GL(V)$, where V is a finite-dimensional vector space with a nondegenerate symplectic form on it (see [EG] for more details). We also have $\beta_V \equiv 0$.

5.8. **Infinitesimal Hecke algebras.** These are defined in [EGG, §4]. Here, $A = \mathfrak{U}\mathfrak{g}$ for $\mathfrak{g}$ the Lie algebra of a reductive algebraic group G (over $\mathbb{C}$), with V an algebraic representation of G. (The unit in $\mathfrak{U}\mathfrak{g} \subset \mathcal{O}(G)^*$ is δ_1.) Once again, $\beta_V \equiv 0$ and $\beta_A \equiv \kappa$. (Also note that for $X \in \mathfrak{g}$ and $v \in V$, we have $Xv = X(v) \cdot \delta_1 + vX$.) The κ's satisfying the PBW property were classified in [EGG] when $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sp}_{2n}$.

5.9. **Symplectic oscillator algebras.** These were defined in [Kh1]: we again have $\dim_R V = 2$ (so Jacobi holds), but with $A = \mathfrak{U}(\mathfrak{sl}_2)$. Now note that $[y,x]$ is an $\mathfrak{h}$-weight vector (where $\mathfrak{h} \subset \mathfrak{sl}_2$) of $\mathfrak{h}$-weight 0. So if we want the PBW property to hold, then the A-equivariance of β suggests that $\operatorname{im} \beta$ is killed by $\operatorname{ad} \mathfrak{h}$ as well, whence $\operatorname{im} \beta_V = 0$. Moreover, $[y,x]$ is a polynomial in the Casimir element, since it must be central in A:

$V \wedge V$ is an $\mathfrak{sl}_2$-module, so it is the trivial (i.e. "one-dimensional") representation. Hence $y \wedge x$ is an $\mathfrak{sl}_2$-weight vector of weight $\varepsilon = 0$. (Alternatively, one can verify that $\langle X(v), v' \rangle = \langle v, -X(v') \rangle$ for $X \in \mathfrak{sl}_2,\ v, v' \in V$, in order to apply Proposition 4.4 - with $M = A = \mathfrak{U}(\mathfrak{sl}_2),\ \beta_V \equiv 0$, and $B = \mathscr{H}_\beta$.)

Using either of these, $\mathscr{H}_\beta$ has the PBW property if and only if (since Jacobi automatically holds) $\beta(y, x)$ is also an ε-weight vector in A (under the adjoint map). By Lemma 4.2, this is if and only if it is central - which, in this case, means a polynomial in the Casimir element.

We finally note that we get a Hopf algebra if and only if $\beta(x, y) = 0$ (i.e. $[x, y] = 0$), since no nonzero polynomial in the Casimir is primitive.

5.10. **Degenerate affine Hecke algebras.** Given a finite-dimensional reductive complex Lie algebra $\mathfrak{g}$, let W be its Weyl group and $\mathfrak{h}$ a fixed chosen Cartan subalgebra. Thus $\mathfrak{h} = \oplus_{i \geq 0} \mathfrak{h}_i$, where for $i > 0$, $\mathfrak{h}_i$ corresponds to a simple component (ideal) of $\mathfrak{g}$, with corresponding base of simple roots Δ_i and Weyl group W_i (so $\Delta = \coprod_{i>0} \Delta_i$ and $W = \times_{i>0} W_i$); and $\mathfrak{h}_0$ is the central ideal in $\mathfrak{g}$.

We define $Q_i = \oplus_{\alpha \in \Delta_i} \mathbb{Z}\alpha$, the root lattice inside $\mathfrak{h}_i^*$, and choose and fix some $\mathbb{Z}$-lattice Q_0 inside $\mathfrak{h}_0^*$. Now define $V = \oplus_{i \geq 0} V_i$, where $V_i := R \otimes_{\mathbb{Z}} Q_i$; and form the algebra $\mathscr{H}_0 = RW \ltimes \operatorname{Sym}_R V$. This is the *degenerate affine Hecke algebra with trivial parameter* (the parameter is trivial since $wv - w(v)w$ is always zero), of reductive type. This is a special case of [Ch, Definition 1.1], where we set $\eta = 0$. As seen above, this is a Hopf algebra with the PBW property.

6. Symplectic reflections

We conclude by elaborating a bit more on what "symplectic reflections" in [EG] generalize to, in this setup. The following result is similar to Theorem 3.4 above. It was first stated in [Dr], and is also shown in [EG, Gr].

Proposition 6.1. *Say A is a cocommutative bialgebra, with comultiplication Δ and counit ε. Suppose $\beta = \beta_A$, and $\mathscr{H}_\beta$ has the PBW property. Given $a' \in A$, suppose there exists (nonzero) $a'' \in A$ and a complement U to Ra'' in A (i.e. $Ra'' \oplus U \subset A$), so that*

$$\Delta(\operatorname{im} \beta) \subset R(a' \otimes a'') \oplus (A \otimes U)$$

but $\Delta(\operatorname{im}(\beta)) \not\subset A \otimes U$. Then $\dim_R \operatorname{im}(a' - \varepsilon(a')) \leq 2$.

(The $\not\subset$ simply means that $a' \otimes a''$ does occur as a component in some $\beta(x, y)$; in particular, $a' \neq 0$.)

Examples.

(1) For example, if $A = RW$ is a group ring and $a' = g \in W$, then choose $U := \sum_{g' \neq g} Rg'$; this is what happens in [Dr, EG, Gr].

(2) Another example would be if $A = \mathfrak{U}\mathfrak{g}$ and $\operatorname{im}\beta \subset \mathfrak{g}$. Then we could take $a' = 1$ and $a'' = X \in \mathfrak{g}$, since $1 \otimes X$ satisfies the condition. However, this leads to a trivial statement, since $\operatorname{im}(1-1) = 0$.

(3) If $\operatorname{im}\beta$ is one-dimensional, then the condition holds for *every* $a_{(1)}$ in the expansion of $\Delta(\beta(x,y))$.

(4) Let us give an example of this last case. Given a group G with a character $\chi : G \to R^\times$, G acts on a free rank-one R-module $R \cdot X$ via χ. Then G acts naturally on $R[X]$. Given a central $g \in \mathfrak{Z}(G)$, define the smash product (Hopf) algebra $H_g := RG \ltimes R[X]$ with the usual operations on RG, and

$$\Delta(X) := g \otimes X + X \otimes g, \qquad \varepsilon(X) = 0, \qquad S(X) := -\chi(g)g^{-2}X$$

This is then a cocommutative Hopf algebra. Now, if $\beta = \beta_A : V \wedge_R V \to R \cdot X$, then $\operatorname{im} X$ and $\operatorname{im}(1-g)$ are at most "two-dimensional subspaces" in V.

Proof. Suppose $a' = a_{(1)}$ occurs while expanding $a = \beta_A(x,y) = \beta(x,y)$ (as the summand $a' \otimes a''$). Now, the Jacobi identity says that

$$[\beta(x,y),z] + [\beta(y,z),x] + [\beta(z,x),y] = 0$$

for all $z \in V$. In other words, we can write the last two commutators in terms of

$$\begin{aligned} \sum a_{(1)}(z)a_{(2)} - za &= \sum (a_{(1)}(z)a_{(2)} - z\varepsilon(a_{(1)})a_{(2)}) \\ &= \sum (a_{(1)} - \varepsilon(a_{(1)}))(z)a_{(2)} \in V \otimes A \hookrightarrow \mathscr{H}_\beta \end{aligned}$$

Now write this sum using $Ra'' \oplus U$ for the second coordinate, and fix one of the summands: $a_{(2)} = a''$. If $a' - \varepsilon(a') = 0$ then we are trivially done; otherwise write out the last two commutators (after moving them to the other side). Thus, if $\beta(z,y) = b, \beta(x,z) = c$, then

$$\sum (a_{(1)} - \varepsilon(a_{(1)}))(z)a_{(2)} = \sum (b_{(1)} - \varepsilon(b_{(1)}))(x)b_{(2)} + \sum (c_{(1)} - \varepsilon(c_{(1)}))(y)c_{(2)}$$

by skew-symmetry of β. We now isolate the $(\otimes a'')$-component on both sides (because $\mathscr{H}_\beta$ has the PBW property). Then by assumption, we must have (in the R-vector space $A \otimes A$):

$$(a' - \varepsilon(a'))(z) \in Rx' + Ry' \ \forall z \in V$$

where $x' = (a' - \varepsilon(a'))(x)$ and $y' = (a' - \varepsilon(a'))(y)$. ∎

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